ELLIPTIC ALGEBRAS AND EQUIVARIANT ELLIPTIC COHOMOLOGY I.

(*technical report*)

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The motivation for writing this paper was the parallelism in the classification of three different kinds of mathematical objects:

(i) Classical r-matrices.

(ii) Generalized cohomology theories that have Chern classes for complex vector bundles.

(iii) 1-dimensional formal groups.

Recall first, that there are some distinguished 1-dimensional formal groups - those corresponding to actual algebraic groups - that is, the additive group $\mathbf{G}_a$, the multiplicative group $\mathbf{G}_m$, and the elliptic curves. Further, classical r-matrices, that is solutions to the classical Yang-Baxter equation

$$[r^{13}(u+v), r^{23}(v) - r^{12}(u)] = [r^{23}(v), r^{12}(u)]. \tag{0.1}$$

were classified by Belavin and Drinfeld [BD]. They showed that , under appropriate non-degeneracy conditions, all algebraic solutions to (0.1) are given by rational, trigonometric and elliptic functions respectively. The "quantum algebras" associated to those solutions are known, respectively, as Yangians, quantized enveloping algebras, and quantum elliptic algebras [Dr1] [RS]. Thus, one gets an 'a posteriori' correspondence (i)↔(iii).

The relation (ii)↔(iii), on the other hand, is well known in algebraic topology, due to Quillen [Q]. The generalized cohomology theories associated to formal groups arising from $\mathbf{G}_a$, $\mathbf{G}_m$ and elliptic curves are, respectively, the ordinary cohomology, the complex K-theory and the so-called elliptic cohomology [Lan][S2].

This suggests looking for a direct construction relating one of the three generalized cohomology theories above with the corresponding quantum algebras. For the case of K-theory such a relation was found in [GV1] following an earlier work [BLM]. It provides a description of a quantized universal enveloping algebra in terms of equivariant K-theory of flag varieties. It was also observed in [GV1] that replacing equivariant K-theory by equivariant homology yields a construction of Yangians.

The aim of the present paper is to treat the elliptic case. The first obstacle here is the absence of a 'good' equivariant elliptic cohomology theory. The non-equivariant elliptic cohomology theory was first introduced in mid 80's, see [Lan] and references therein, though no direct *geometric* interpretation of that theory is known as yet. Recently, a definition of equivariant elliptic cohomology with complex coefficients was suggested by Grojnowski [Gr]. His approach is not fully satisfactory however, since the construction in [Gr] becomes void in the non-equivariant case: it cannot distinguish non-equivariant elliptic cohomology from the ordinary cohomology at all.

In the present paper we adopt the following strategy. In the first two sections we work out general formalism of the 'would-be' equivariant elliptic cohomology theory as if such a theory existed. Most of our attention is payed to those features of the theory that are essentially different from the known cohomology

theories, e.g., an invariant construction of the Chern classes. At the end of §1 we sketch Grojnowski's definition (we slightly modify his definition actually in order to remain in the domain of algebraic, rather than complex-analytic, geometry). Our exposition is thus very similar, in spirit, to the Kazhdan-Lusztig exposition [KL], in the Hecke algebra case. There, basic properties of the equivariant K-homology theory were listed in detail while the definition itself was only sketched. The reason for this is the same both in [KL] and in our paper: the information about the cohomology theory under consideration used in the study of elliptic, resp. Hecke, algebras does not depend in any way on the definition of the cohomology theory itself.

Our new results are concentrated mostly in chapters 3-5. The heart of the paper is the algebro-geometric construction of n. 3.4, which is quite general and does not depend on the rest of the material. The main results of the parer are theorem 3.5, and theorem 5.13. Proof of proposition 3.6 is technically most complicated; it requires a refinement of a lengthy argument used in [BLM].

Part 5 provides also a self-contained exposition of elliptic algebras that might be useful for the non-expert. The material involved does not seem to be well covered in the literature.

The present paper should be regarded as a preliminary report on a wider project. This project involves deeper understanding of the recently discovered relationship between loop and 'double-loop' groups on the one hand, and moduli spaces of algebraic vector bundles on an elliptic curve and or an elliptic surface, on the other (cf. [EFK] and [GKV]). The subject is also closely related to the construction of 'Springer resolutions' and theory of 'Springer representations' for loop groups. These matters will be discussed in more detail in a future publication.

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1. Equivariant elliptic cohomology : axiomatics.

(1.2) Elliptic curves. Let S be a scheme. By an elliptic curve over S we mean a group scheme $p : E \to S$ over S whose fiber at every geometric point is an elliptic curve. Thus E is equipped with the zero section $i : S \to E$. We denote the line bundle $i^*\Omega^1_{E/S}$ on S by ω_E or simply ω. We denote by $E^\vee \to S$ the dual elliptic curve parametrizing line bundles on X of relative degree 0. Note that $\omega_{E^\vee} = (\omega_E)^*$ and $E^{\vee\vee} = E$. We denote by E^r and $E^{(r)}$ respectively the Cartesian and symmetric powers of E over S.

(1.3) The space $\mathcal{X}_G$. Let G be a compact Lie group. Then G is a maximal compact subgroup in a uniquely determined complex algebraic Lie group $G_{\mathbf{C}}$, and the categories of continuous finite-dimensional representations of G and algebraic representations of $G_{\mathbf{C}}$ are known to be equivalent. Moreover, it was shown by Chevalley that $G_{\mathbf{C}}$ comes in fact from a group scheme G_{alg} over $\mathbf{Z}$. For example, if $G = U(n)$ is the unitary group, then $G_{\mathbf{C}} = GL_n(\mathbf{C})$ and G_{alg} is the group scheme GL_n. If G is a finite group, then $G_{alg} = G_{\mathbf{C}} = G$. We write G^0 for the identity component of G, and the same for $G_{\mathbf{C}}$, G_{alg}.

Let $p : E \to S$ be an elliptic curve. A principal G_{alg}-bundle P over E defines a finite covering $\tilde{E} \to E$ consisting of "connected components of the fibers". The canonical morphism $P \to \tilde{E}$ makes P a principal G^0_{alg}-bundle over $\tilde{E}$.

Let $\mathcal{X}_G$ be the moduli scheme of semistable principal G-bundles P over $E^\vee$ (see [R]) such that $P \to \tilde{E}$ is topologically trivial on each geometric fiber. This is a smooth scheme over S, and we let $p_G : \mathcal{X}_G \to S$ denote the canonical projection. We write $0 \subset \mathcal{X}_G$ for the "zero section" given by the class of trivial G-bundle. A homomorphism $\phi : G \to H$ of compact Lie groups induces a morphism of groups schemes $G_{alg} \to H_{alg}$ and thus a morphism of S-schemes $\mathcal{X}_\phi : \mathcal{X}_G \to \mathcal{X}_H$.

We writeω for the line bundle $p_G^*\omega$ on $\mathcal{X}_G$, for short.

(1.4) Examples.
(1.4.1) If $G = S^1$ is the circle group, then $\mathcal{X}_G = E^{\vee\vee} = E$.

(1.4.2) If G is a torus, then let $\Gamma = \mathrm{Hom}(S^1, G)$ be the lattice of 1-parameter subgroups in G, so $G = S^1 \otimes_{\mathbf{Z}} \Gamma$. Then $\mathcal{X}_G = E \otimes_{\mathbf{Z}} \Gamma$ is the sum of $\mathrm{rk}(\Gamma)$ copies of E.
(1.4.3) If $G = \mathbf{Z}/n$, then $\mathcal{X}_G = E_n \subset E$ is the subgroup of n-torsion points of E.
(1.4.4) The above examples can be unified as follows. Let G be a compact Abelian group. Then the Pontryagin dual $\hat{G} = \mathrm{Hom}(G, S^1)$ is a discrete Abelian group, and $\mathcal{X}_G = \mathrm{Hom}(\hat{G}, E)$. In particular, $\mathcal{X}_G$ is naturally endowed with a structure of compact Abelian group. Indeed, given an isomorphism of Lie groups $E \simeq S^1 \times S^1$, the Lie group $\mathcal{X}_G$ is naturally isomorphic to $G \times G$ in such a way that for any closed subgroup $H \subset G$ the closed subgroup $\mathcal{X}_H \subset \mathcal{X}_G$ is identified with $H \times H \subset G \times G$.
(1.4.5) If G is a connected compact Lie group with maximal torus T and Weyl group W, then $\mathcal{X}_G = \mathcal{X}_T/W$. For example, if $G = U(n)$, then $\mathcal{X}_G = E^{(n)}$ is the nth symmetric product of X (over S). If $G = SU(n)$, then $\mathcal{X}_G$ is the kernel of the addition map $E^{(n)} \xrightarrow{+} E$. This is a $\mathbf{P}^{n-1}$-bundle over S. More generally, if G is a simple Lie group, then, as shown by Bernstein and Schwartzman [BS], the fibers of $p_G : \mathcal{X}_G \to S$ are weighted projective spaces. The space $\mathcal{X}_G$ was also studied in [Lo].
(1.4.6) Let $S = \mathrm{Spec}(\mathbf{C})$, so E is an elliptic curve over $\mathbf{C}$. Then, by Weil - Narasimhan - Seshadri theorem $\mathcal{X}_G$ is the same as the moduli spaces of principal G-bundles over E with an integrable connection, i.e., of conjugacy classes of homomorphisms $\pi_1(E) \to G$. Since $\pi_1(E) = \mathbf{Z}^2$, we find that $\mathcal{X}_G$ is the set of conjugacy classes of pairs of commuting elements of G. This is to be compared with [S2], [HKR].

(1.5) Equivariant elliptic cohomology.

For every scheme Z we denote by $\mathrm{Coh}(Z)$ the category of coherent sheaves of $\mathcal{O}_Z$ - modules.

Let G be a compact Lie group By a G-cell we mean, (see [W]), a G-space of the form $(G/H) \times D^n$ where H is a closed subgroup in G and D^n is a disk of dimension n with trivial G-action. A G-CW complex (G-complex, for short) is a G-space M glued from G-cells. Thus M admits an increasing filtration M_n by G-subspaces such that M_n/M_{n-1} is a bouquet of G-spaces of the form $(G/H) \times S^n$, where S^n is the n-sphere. It is known that any smooth manifold with smooth G-action is a finite G-complex, see [AP].

In this paper we choose a naive point of view on (equivariant) cohomology theories as functors on pairs of G-complexes, close to that of tom Dieck [Di]. Let $E \to S$ be an elliptic curve. A G-equivariant elliptic cohomology theory associated to E is a collection of the following data :

(1.5.1) *Contravariant* functors Ell^i_G $(i = 0, 1, \ldots)$ from the category of pairs of finite G-complexes into $\mathrm{Coh}(\mathcal{X}_G)$.

(1.5.2) *Natural* sheaf morphisms $\partial : \mathrm{Ell}^i_G(A) \to \mathrm{Ell}^{i+1}_G(M, A)$ given for any pair (M, A).

(1.5.3) *Multiplication maps* $\mathrm{Ell}^i_G(M, A) \otimes \mathrm{Ell}^j_G(N, B) \to \mathrm{Ell}^{i+j}_G(M \times N, M \times B \cup A \times N)$ which are associative, graded commutative and functorial in an obvious sense.

Put $\mathrm{Ell}_G = \bigoplus_i \mathrm{Ell}^i_G$. The data (1.4.1-3) should satisfy the following conditions :

(1.5.4) *Homotopy and exactness :* Two G-homotopic maps of pairs induce the same maps on Ell^i_G, and for every pair (M, A) the following natural sequence is exact

$$\ldots \to \mathrm{Ell}^i_G(M, A) \to \mathrm{Ell}^i_G(M) \to \mathrm{Ell}^i_G(A) \xrightarrow{\partial} \mathrm{Ell}^{i+1}_G(M, A) \to \ldots$$

(1.5.5) *Periodicity axiom :* There are natural isomorphisms

$$\mathrm{Ell}^{i-2}_G(M, A) \simeq \mathrm{Ell}^i_G(M, A) \otimes \omega. \quad \text{Moreover,} \quad \mathrm{Ell}^{2i+1}_G(pt) = 0, \quad \mathrm{Ell}^{2i}_G(pt) = \omega^{\otimes(-i)}.$$

Note that for the case $G = \{1\}$, $S =$ modular curve, (1.5.5) gives that the space of global section of $\mathrm{Ell}^{-2i}_1(pt)$ is the space of modular forms of weight i. Note also that (1.5.4) implies that Ell_G satisfies the "suspension axiom" : $\mathrm{Ell}^i_G(\Sigma M, \Sigma A) = \mathrm{Ell}^{i-1}_G(M, A)$, where Σ denotes suspension.

(1.6) Relations between different groups. We now list axioms relating theories Ell_G for different G.

(1.6.1) For any homomorphism of compact Lie groups $\phi : G \to H$, there is a multiplicative morphism of functors $T_\phi : \mathrm{Ell}_H \Rightarrow \mathcal{X}_{\phi_*}\mathrm{Ell}_G$ from the category of H-pairs of complex to $\mathrm{Coh}(\mathcal{X}_H)$. For any two composable homomorphisms ϕ, ψ we have $T_{\psi\phi} = T_\phi \circ T_\psi$.

(1.6.2) *Change of groups :* If $\phi : G \to H$ is a group homomorphism, then for any H-complex M we have an Eilenberg-Moore spectral sequence

$$E_1^{i,j} = L_i \mathcal{X}_\phi{}^* \mathrm{Ell}_H^j(M) \Rightarrow \mathrm{Ell}_G^{j-i}(M).$$

Here $L_i \mathcal{X}_\phi{}^*$ is the i-th left derived functor of the inverse image (This sequence converges, since $\mathcal{X}_G$ has finite Tor-dimension over $\mathcal{X}_H$). In particular,

(1.6.3) If $\mathcal{X}_\phi$ is a flat morphism, there is an isomorphism $\mathcal{X}_\phi^* \mathrm{Ell}_H \xrightarrow{\sim} \mathrm{Ell}_G$.

(1.6.4) *Induction axiom :* Let $H \subset G$ be the embedding of a closed normal subgroup, and M be a G-complex such that the action of H is free. Denote by $p : M \to M/H$ and $\phi : G \to G/H$ the projections. Then the map $p^* \circ T_\phi(M/H) : \mathrm{Ell}_{G/H}(M/H) \to \mathcal{X}_{\phi_*} \mathrm{Ell}_G(M)$ is an isomorphism of sheaves.

(1.6.5) *Künneth formula :* Let G, H be two compact Lie group, M be a G-complex and N be an H-complex. Then $\mathrm{Ell}_{G \times H}(M \times N) = \mathcal{X}_{\rho_G}^* \mathrm{Ell}_G(M) \otimes \mathcal{X}_{\rho_H}^* \mathrm{Ell}_H(N)$, where ρ_G, ρ_H are the projections of $G \times H$ to G, H.

We conjecture that any elliptic curve E gives rise to a unique equivariant elliptic cohomology theory, natural in E. A more systematic, if more technical, way of formulating this conjecture is by using the language of spectra representing cohomology theories. Namely, for any G there should exist a canonical sheaf $\mathcal{F}_G$ of G-spectra (in the sense of [LMS]) on $\mathcal{X}_G$, and for any homomorphism $\phi : G \to H$ we should have $\mathcal{F}_G = (L\phi^*)\mathcal{F}_H$ where $L\phi^*$ is the derived functor of the inverse image. For $G = \{1\}$ this would give a sheaf of spectra on S considered in [H]. In Section 2 we will give a construction valid over the rationals.

(1.7) Some consequences of the definition.

(1.7.1) For a locally compact space M let $M \cup \infty$ denote the one-point compactification of M. Then (1.5.4-5) implies that (for the trivial G-action) $\mathrm{Ell}_G^0(\mathbf{R}^n \cup \infty) = 0$ for odd n while $\mathrm{Ell}_G^0(\mathbf{C}^n \cup \infty) = \omega^{\otimes(n)}$.

(1.7.2) Let 1 denote the group with one element. By applying the induction axiom (1.6.1) to finite-dimensional G-invariant skeletons of the contractible space EG, we find that $\mathrm{Ell}_1^0(BG)$ is the completion of the sheaf $\mathcal{O}_{\mathcal{X}_G}$ at the point $0 \in \mathcal{X}_G$. In particular, taking $G = S^1$, we find that $\mathrm{Ell}_1^0(\mathbf{CP}^\infty) := \lim \mathrm{inv}\, \mathrm{Ell}_1^0(\mathbf{CP}^n)$ is the completion of $\mathcal{O}_E$ at $0 \in E$.

(1.7.3) Proposition. *Let $\phi : H \hookrightarrow G$ be the embedding of a closed subgroup of G. Then $\mathcal{X}_{\phi_*} \mathrm{Ell}_H(pt)$ and $\mathrm{Ell}_G(G/H)$ are isomorphic coherent sheaves on $\mathcal{X}_G$.*

Proof. Let $\delta : H \hookrightarrow G \times H$ be the diagonal map and ρ_G, ρ_H be the projections of $G \times H$ to G, H. Let $\{1\} \xrightarrow{i} G \xrightarrow{p} \{1\}$ be the canonical maps. Since $p^* \circ T_{\rho_H}(pt) : \mathrm{Ell}_H(pt) \to \mathcal{X}_{\rho_H *} \mathrm{Ell}_{G \times H}(G)$ is an isomorphism by (1.6.4), the map $i^* \circ T_\delta(G) : \mathrm{Ell}_{G \times H}(G) \to \mathcal{X}_{\delta *} \mathrm{Ell}_H(pt)$ is either an isomorphism. Composing it with the isomorphism $\mathrm{Ell}_G(G/H) \to \mathcal{X}_{\rho_G *} \mathrm{Ell}_{G \times H}(G)$ given by (1.6.4), one gets the result.

(1.7.4) The scheme $M_{\mathcal{X}_G}$. By (1.5.3), any $\mathrm{Ell}_G^0(M)$ is a sheaf of commutative algebras over $\mathcal{X}_G$. We denote its spectrum by $M_{\mathcal{X}_G}$ and denote by $\pi_M : M_{\mathcal{X}_G} \to \mathcal{X}_G$ the projection. In particular, $\mathrm{Ell}_G^0(M) = \pi_{M*} \mathcal{O}_{M_{\mathcal{X}_G}}$ and, if $M = \{pt\}$, we get $pt_{\mathcal{X}_G} = \mathcal{X}_G$. If $\phi : M \to N$ is a holomorphic map the contravariant functoriality map $\phi^* : \mathrm{Ell}_G(N) \to \mathrm{Ell}_G(M)$ induces a morphism of schemes $\phi_{\mathcal{X}_G} : M_{\mathcal{X}_G} \to N_{\mathcal{X}_G}$. The maps $M \mapsto M_{\mathcal{X}_G}$, $\phi \mapsto \phi_{\mathcal{X}_G}$ define a functor from the category of G-complexes to the category of schemes over $\mathcal{X}_G$. Note that, from Proposition (1.7.3), if M is a finite G-complex then π_M is a finite morphism. Indeed Proposition (1.7.3) can be rephrased in the following way :

(1.7.5). *Let $\phi : H \hookrightarrow G$ be the embedding of a closed subgroup of G. Then $(\mathcal{X}_H, \mathcal{X}_\phi)$ and $((G/H)_{\mathcal{X}_G}, \pi_{G/H})$ are isomorphic schemes over $\mathcal{X}_G$.*

Section (1.6) can be rewritten in terms of $M_{\mathcal{X}_G}$. Axioms (1.6.1) and (1.6.3) mean that a homomorphism $\phi : G \to H$ and a H-complex M give a commutative square

$$\begin{array}{ccccc} & M_{\mathcal{X}_G} & \longrightarrow & M_{\mathcal{X}_H} & \\ \pi_M & \downarrow & & \downarrow & \pi_M \\ & \mathcal{X}_G & \xrightarrow{\mathcal{X}_\phi} & \mathcal{X}_H & \end{array}$$

which is Cartesian if $\mathcal{X}_\phi$ is flat. Similarly, (1.6.4) means that if a normal subgroup H of G acts freely on a G-complex M then the quotient map $\phi : G \to G/H$ induces a commutative diagram

$$\begin{array}{ccccc} & M_{\mathcal{X}_G} & \xrightarrow{=} & (M/H)_{\mathcal{X}_{G/H}} & \\ \pi_{_M} & \downarrow & & \downarrow & \pi_{_{M/H}} \\ & \mathcal{X}_G & \xrightarrow{\mathcal{X}_\phi} & \mathcal{X}_{G/H} & . \end{array}$$

Formally, $M_{\mathcal{X}_G}$, $\mathcal{X}_G$ and $\pi_{_M}$ behave exactly in the same way that M_G, BG and the canonical fibration $M_G \to BG$ do in a completed theory.

(1.7.6) For any triple (M, A, B), $B \subset A \subset M$, of finite G-complexes (1.5.4) and functoriality give an exact sequence

$$... \to \mathrm{Ell}^i_G(M, A) \to \mathrm{Ell}^i_G(M, B) \to \mathrm{Ell}^i_G(A, B) \xrightarrow{\partial} \mathrm{Ell}^{i+1}_G(M, A) \to ...$$

(1.8) Chern classes. If H is another Lie group and $P \to M$ is a G-equivariant principal H-bundle over a G-space M, then (1.6.1) gives a regular map

$$c_P : M_{\mathcal{X}_G} = P_{\mathcal{X}_{G\times H}} \to \mathcal{X}_{G\times H} \to \mathcal{X}_H.$$

We call this map the *characteristic class* of P. In particular, if V is a G-equivariant vector bundle over M of rank r with Hermitian form in the fibers, then the principal $U(r)$-bundle of orthonormal frames in V gives a map $c_V : M_{\mathcal{X}_G} \to E^{(r)}$. It is clear that

$$c_{V\oplus W} = c_V \oplus c_W, \quad c_{V\otimes W} = c_V \otimes c_W,$$

where $\oplus$ and $\otimes$ on the right hand sides of the equalities are defined as follows :

$$\oplus : E^{(r_1)} \times E^{(r_2)} \to E^{(r_1+r_2)}, \quad \text{the} \quad \text{symmetrization} \quad \text{map},$$

$$\otimes : E^{(r_1)} \times E^{(r_2)} \to E^{(r_1 r_2)}, \quad (\{x_1, ..., x_{r_1}\}, \{y_1, ..., y_{r_2}\}) \mapsto \{x_i + y_j\}.$$

By a *coordinate* on E we mean a rational section f of the line bundle $p^*\omega^{-1}$ (where $p : E \to S$ is the projection) with the following property:

(1.8.1). *The section f is regular near $0 \subset E$ and the differential $d_{E/S}(f)$, being restricted to 0, coincides with the identity map $\omega^{-1} \to \omega^{-1}$.*

Thus, on each geometric fiber $E_s, s \in S$, a coordinate f gives a rational function but with values in the tangent space T_0E_s (very much like the logarithm in a Lie group). If E is equipped with a coordinate, then for any G-equivariant vector bundle V on a G-space M and any $i \geq 0$ we have its equivariant Chern class $c_i^f(V)$ which is a rational section of the sheaf $\mathrm{Ell}^{2i}_G(M)$ regular near 0. The class $c_i^f(V)$ is described as follows. Namely, let $p_r : E^{(r)} \to S$ be the projection and $\sigma_i(f)$ be the rational section of the bundle $p_r^*\omega^{\otimes(-i)}$ on $E^{(r)}$ given by $\sigma_i(f)(x_1, ..., x_r) = \sigma_i(f(x_1), ..., f(x_r))$, where σ_i is the ith elementary symmetric function. We define c_i^f to be the pullback of the $\sigma_i(f)$ with respect to the map $c_V : M_{\mathcal{X}_G} \to E^{(r)}$. This is a section of the bundle $\pi_M^*\omega^{\otimes(-i)}$ on $M_{\mathcal{X}_G}$, i.e., a section of $\mathrm{Ell}^{2i}_G(M)$ on $\mathcal{X}_G$. It is clear that the Chern classes thus defined satisfy the usual properties. In particular, it follows that the non-equivariant theory Ell_1 becomes complex oriented. The corresponding formal group law is the series

$$F(x, y) = x + y + \sum_{i+j\geq 2} a_{ij}x^iy^j, \quad a_{ij} \in H^0(S, \omega^{\otimes i+j-1}) = \mathrm{Ell}_1^{-2i-2j+2}(pt),$$

giving the addition theorem for f, i.e. $f(u + v) = F(f(u), f(v))$.

A similar construction with f a section of a non-trivial line bundle on E gives the Euler class (see (2.6)).

(1.8.2) Example. Let $M = pt$ and $V = \mathbf{C}^r$ be a vector space, $G = U(1)^n$ be a torus acting on V via characters $\theta_1, ..., \theta_r$. Let $\mathcal{X}_{\theta_i} : E^n \to E$ be the morphism of algebraic groups given by the exponents entering into θ_i. Then

$$c_V = \mathcal{X}_{\theta_1} \oplus \mathcal{X}_{\theta_2} \oplus ... \oplus \mathcal{X}_{\theta_r} : E^n \to E^{(r)}.$$

(1.8.3) Remarks.
(1.8.3.1) The formal germ at 0 of a coordinate f gives an element of $\mathrm{Ell}_1^2(\mathbf{CP}^\infty)$ (just recall that $\mathrm{Ell}_1^0(\mathbf{CP}^\infty)$ is the completion of $\mathcal{O}_E$ at $0 \in E$) and the differential $d_{E/S}f|_0$ along 0 corresponds to the image of this element in $\mathrm{Ell}_1^2(\mathbf{CP}^1) = \mathrm{Ell}_1^0(pt) = \mathcal{O}_S$. So the above choice of the notion of coordinate exactly corresponds to the concept of a complex orientation in topology, as explained in [AHS].
(1.8.3.2) There are several versions of elliptic cohomology considered in the literature, for instance, "classical" elliptic cohomology related to the Jacobi sine and points of order 2, Hirzebruch's level N elliptic cohomology etc, see [Lan], [HBJ]. From our point of view, the difference among them is the artefact of the choice of coordinate, which is made to pass from a formal group (a group object in the category of formal schemes) to a formal group law (a power series $F(x, y)$). Namely, an elliptic curve E by itself does not have any preferred coordinate and so does not define any formal group law. One natural choice of a coordinate is associated to a point $\eta \in E$ of order 2. This is the function (Jacobi sine) on E having simple zeroes at 0 and η and simple poles at the two other points of second order. This choice of coordinate leads to the level 2 elliptic cohomology. In a similar way, a surjective homomorphism $\beta : E_N \to \mathbf{Z}/N$, where E_N is the set of N-torsion points on E, gives a coordinate f with simple zeroes on $\beta^{-1}(0)$ and simple poles on $\beta^{-1}(1)$, and this choice of f leads to the level N elliptic cohomology of Hirzebruch [HBJ]. In our approach there is exactly one elliptic cohomology theory associated to a given elliptic curve.

(1.9) Cohomology of projective and flag bundles. Let M be a finite G-complex and V be a G-vector bundle on M, of rank r. We can assume that V is equipped with an Hermitian metric in the fibers. Let $\mathrm{Fr}(V)$ be the space of orthonormal frames in V. It is acted upon by $G \times U(r)$. The projectivization $\mathbf{P}(V)$ is the quotient $\mathrm{Fr}(V)/U(1) \times U(r-1)$. Thus (1.6.4) gives

(1.9.1) $$\mathbf{P}(V)_{\mathcal{X}_G} = \mathrm{Fr}(V)_{\mathcal{X}_{G\times U(1)\times U(r-1)}} \quad , \quad M_{\mathcal{X}_G} = \mathrm{Fr}(V)_{\mathcal{X}_{G\times U(r)}}.$$

Consider the embedding $\phi : G \times U(1) \times U(r-1) \hookrightarrow G \times U(r)$. From (1.7.5), the corresponding map $\mathcal{X}_\phi : \mathcal{X}_{G\times U(1)\times U(r-1)} \to \mathcal{X}_{G\times U(r)}$ is just the symmetrization map $\mathcal{X}_G \times E \times E^{(r-1)} \to \mathcal{X}_G \times E^{(r)}$. This map is flat and therefore

(1.9.2) $$\mathbf{P}(V)_{\mathcal{X}_G} = \mathrm{Fr}(V)_{\mathcal{X}_{G\times U(1)\times U(r-1)}} = \mathrm{Fr}(V)_{\mathcal{X}_{G\times U(r)}} \times_{\mathcal{X}_{G\times U(r)}} \mathcal{X}_{G\times U(1)\times U(r-1)}.$$

Thus, since $\mathrm{Fr}(V)_{\mathcal{X}_{G\times U(r)}} = M_{\mathcal{X}_G}$, we get the following Cartesian square

(1.9.3) $$\begin{array}{ccc} \mathbf{P}(V)_{\mathcal{X}_G} & \longrightarrow & E \times E^{(r-1)} \\ \downarrow & & \downarrow \\ M_{\mathcal{X}_G} & \xrightarrow{c_V} & E^{(r)} \end{array} .$$

Note that the map $\mathbf{P}(V)_{\mathcal{X}_G} \to M_{\mathcal{X}_G} \times E$ is an embedding.

If we choose a coordinate f on E (by shrinking S, if necessary), then we get Chern classes $c_i^f(V) \in \Gamma_{rat}(\mathrm{Ell}_G^{2i}(M))$ and (1.9.3) yields, after passing to the completion at 0, the standard description of the (completed) equivariant elliptic cohomology of $\mathbf{P}(V)$ in terms of Chern classes.

In a similar way one can treat general flag varieties. Let $\mathbf{r} = (r_1, ..., r_n)$ be a positive integer vector such that $\sum r_i = r$, and let $F_{\mathbf{r}}(V)$ be the variety of flags of subspaces $V_1 \subset ... \subset V_n$ in fibers of V such that $\dim(V_i/V_{i-1}) = r_i$. Let $E^{(\mathbf{r})} = \prod E^{(r_i)}$. Then we have a Cartesian square

(1.9.4) $$\begin{array}{ccc} F_{\mathbf{r}}(V)_{\mathcal{X}_G} & \longrightarrow & E^{(\mathbf{r})} \\ \downarrow & & \downarrow \\ M_{\mathcal{X}_G} & \longrightarrow & E^{(r)} \end{array} ,$$

where the right vertical map is the symmetrization.

(1.10) Examples.

(1.10.1) Let $M = pt$ and $V = \mathbf{C}^r$ be a vector space, $G = U(1)^n$ be a torus acting on V via characters $\chi_1, ..., \chi_r$, so that χ_i is taken μ_i times. Then $\mathbf{P}(V)_{\mathcal{X}_G} = \bigcup \Gamma_i^{[\mu_i]}$ where $\Gamma_i \subset E^{n+1}$ is the graph of the homomorphism of algebraic groups $\phi_i : E^n \to E$ given by the exponents entering into χ_i and the superscript μ_i means the μ_ith infinitesimal neighborhood. The natural map

$$\mathbf{P}(V)_{\mathcal{X}_G} = \bigcup \Gamma_i^{[\mu_i]} \to E^n = \mathcal{X}_G$$

is just the natural projection of the graphs of the homomorphisms onto the domain.

(1.10.2) Let $G = U(n)$, and M be the projective space $\mathbf{CP}^{n-1}$ with standard action. Then $\pi_{_M} : M_{\mathcal{X}_G} \to \mathcal{X}_G$ is the canonical projection $E \times E^{(n-1)} \to E^{(n)}$.

(1.10.3) Let $G = U(n)$ and M be the G-manifold of complete flags in $\mathbf{C}^n$ with standard action. Then $\pi_{_M} : M_{\mathcal{X}_G} \to \mathcal{X}_G$ is just the canonical projection $E^n \to E^{(n)}$. Let θ be a character of the torus $H = U(1)^n$ and $L_\theta = G \times_{_H} \mathbf{C}_\theta$ the corresponding line bundle on M. From (1.7.3) and (1.8.2) the characteristic class $c_{L_\theta} : E^n \to E$ is the homomorphism of algebraic groups, $\mathcal{X}_\theta$, given by the exponents entering into θ.

(1.11) Grojnowski's construction. We sketch a construction of equivariant elliptic cohomology, $\mathrm{Ell}_T^0(M)$, for a compact torus $T = S^1 \times \ldots \times S^1$ acting on a a space M. In general, if G is a connected compact group, T a maximal torus in G, and M is a G-variety, we have by (1.4.5) $\mathcal{X}_G = \mathcal{X}_T/W$. Accordingly, we set by definition

$$\mathrm{Ell}_G^0(M) := \mathrm{Ell}_T^0(M)^W,$$

Fix an elliptic curve E, viewed as a 1-dimensional complex Lie group. Let $\exp : \mathrm{Lie}E \to E$ be the (non-algebraic) exponential map. A choice of basis in the lattice Kerexp yields, via the exponential map, an isomorphism $E \simeq \mathbf{C}/(\mathbf{Z} + \tau\mathbf{Z})$. View $\mathbf{C} \simeq \mathrm{Lie}E$ as a 2-dimensional real vector space $\mathbf{R^2}$. Then the isomorphism above gives a real Lie group isomorphism

$$S^1 \times S^1 \xrightarrow{\sim} E \quad , \quad \mathbf{R}/\mathbf{Z} \times \mathbf{R}/\mathbf{Z} \ni (\mathbf{x}, \mathbf{y}) \mapsto \mathbf{x} + \tau \cdot \mathbf{y} \bmod (\mathbf{Z} + \tau\mathbf{Z}) \qquad \mathbf{(1.11.1)}$$

The natural $SL_2(\mathbf{Z})$-action on $\mathbf{R^2}$ induces an action on $S^1 \times S^1 \simeq \mathbf{R}/\mathbf{Z} \times \mathbf{R}/\mathbf{Z}$ by the formula:

$$\binom{a\ b}{c\ d} : (u, v) \mapsto (u^a v^b, u^c v^d) \quad , \quad u, v \in S^1$$

It is immediate from this formula that the subgroup in S^1 generated by u and v depends only on the $SL_2(\mathbf{Z})$-orbit of $(u, v) \in S^1 \times S^1$ and not on a particular representative. Hence, this subgroup is independent of the choice of basis in the lattice Ker(exp). Thus, to any element $e \in E$ we can associate canonically a subgroup in S^1 defined as follows

$$\langle e \rangle = \{u^a \cdot v^b \, , \, a, b \in \mathbf{Z}\}, \qquad (\mathbf{1.11.2}))$$

where the pair (u, v) corresponds e under isomorphism (1.11.1)

Now let T be a compact torus, and $\Gamma = \mathrm{Hom}(S^1, T)$ the lattice of 1-parameter subgroups in T. Then, by (1.4.2) we have canonical isomorphisms

$$T = S^1 \otimes_{\mathbf{Z}} \Gamma \quad , \quad \mathcal{X}_T = E \otimes_{\mathbf{Z}} \Gamma \qquad \mathbf{(1.11.3)}$$

Combining these isomorphisms with (1.11.1) we get a chain of group isomorphisms

$$E \otimes_{\mathbf{Z}} \Gamma \simeq (S^1 \times S^1) \otimes_{\mathbf{Z}} \Gamma \simeq T \times T \qquad \mathbf{(1.11.4)}$$

Let $x \in E \otimes_{\mathbf{Z}} \Gamma$, and let $(g_1, g_2) \in T \times T$ be the pair corresponding to x via the above isomorphism. The trivial, though crucial, observation, based on formula (1.11.2), is that, the subgroup $\langle x \rangle \subset T$ generated by g_1 and g_2 is canonically associated with x, and is independent of the choice of basis in the kernel of $\exp : \mathrm{Lie}E \to E$.

Let M be a T-space. For any pair $g_1, g_2 \in T$ write M^{g_1,g_2} for the set of points of M simultaneously fixed by g_1 and g_2. By the remark in the previous paragraph, this fixed point set is totally determined by the corresponding point $x \in E \otimes_{\mathbf{Z}} \Gamma$, so that we may (and will) write $M^{\langle x \rangle} := M^{g_1,g_2}$

To define $\mathcal{X}_T(M)$ as a *complex analytic* scheme we have, by (1.4.1), to construct a an *analytic* sheaf of algebras on $E \otimes_{\mathbf{Z}} \Gamma$ in the Hausdorff topology. The geometric fiber of this sheaf at a point $\langle x \rangle \in E \otimes_{\mathbf{Z}} \Gamma$ will be the finite dimensional $\mathbf{C}$-algebra $H^*(M^{\langle x \rangle})$, the ordinary cohomology with complex coefficients of the simultaneous fixed point set of the pair $g_1, g_2 \in T$ canonically associated to $x \in \mathrm{Lie}E \otimes_{\mathbf{Z}} \Gamma$.. To glue the fibers together in an analytic sheaf, suffices it to define, for any point $x \in E \otimes_{\mathbf{Z}} \Gamma$ and a small enough open neighborhood $U \subset E \otimes_{\mathbf{Z}} \Gamma$, the space $\Gamma(U, \mathrm{Ell}^0_T(M))$ of its holomorphic sections.

Let $\mathbf{t}_{\mathbf{C}}$ be the complexified Lie algebra of the torus T. By (1.11.3) we have a canonical isomorphism

$$(\mathbf{t}_{\mathbf{C}}) \simeq \mathrm{Lie}E \otimes_{\mathbf{Z}} \Gamma \tag{1.11.5}$$

Fix a point $x \in E \otimes_{\mathbf{Z}} \Gamma$ and its small connected neighborhood U, as in the previous paragraph. Let $\mathcal{U}$ be a connected component of the inverse image of U under the natural exponential map

$$\exp : \mathbf{t}_{\mathbf{C}} = (\mathrm{Lie}E) \otimes_{\mathbf{Z}} \Gamma \to E \otimes_{\mathbf{Z}} \Gamma$$

This map gives an analytic isomorphism $\mathcal{U} \xrightarrow{\exp} \mathcal{U}$, provided U is small enough.

Further, write $H^*_T(-)$ for the T-equivariant cohomology functor. Recall that, for any T-space Y, the equivariant cohomology $H^*_T(Y)$ has a natural $H^*_T(pt)$-module structure. It is known that $H^*_T(pt) = \mathbf{C}[\mathbf{t}_{\mathbf{C}}]$, is the polynomial algebra. We view the polynomial algebra as a subalgebra of the sheaf , $\mathcal{O}^{an}_{\mathbf{t}_{\mathbf{C}}}$, of holomorphic functions on $\mathbf{t}_{\mathbf{C}}$.

Now given a point $x \in E \otimes_{\mathbf{Z}} \Gamma$ and its small connected neighborhood U, following Grojnowski, we set

$$\Gamma(U, \mathrm{Ell}^0_T(M)) := \Gamma(\mathcal{U}, \mathcal{O}^{an}_{\mathbf{t}_{\mathbf{C}}}) \otimes_{\mathbf{C}[\mathbf{t}_{\mathbf{C}}]} H^*_T(M^{\langle x \rangle}),$$

where $M^{\langle x \rangle} := M^{g_1,g_2}$ is the simultaneous fixed point set of the pair $g_1, g_2 \in T$ canonically associated to $x \in (\mathrm{Lie}E) \otimes_{\mathbf{Z}} \Gamma$. The arguments above show that the RHS is well defined and is independent of the choices involved.

2. Thom bundles and Gysin maps.

(2.1) Thom bundles. Let M be a finite G-complex and V be a complex G-vector bundle over M of rank r. The Thom space $\mathrm{Th}(V)$ is the quotient $\mathbf{P}(V \oplus \mathbf{C})/\mathbf{P}(V)$. The relative elliptic cohomology $\mathrm{Ell}^0_G(\mathbf{P}(V \oplus \mathbf{C}), \mathbf{P}(V))$ is a sheaf of modules over $\mathrm{Ell}^0_G(M)$ and thus has the form $\pi_{M*}\Theta(V)$ for a uniquely determined coherent sheaf $\Theta(V)$ on $M_{\mathcal{X}_G}$. This sheaf turns out to be invertible, and the following description of it is immediately deduced from the long exact sequence of cohomology and the description of cohomology of projective bundles given in (1.9).

Consider $\mathbf{P}(V)_{\mathcal{X}_G}$ as a codimension one subscheme in $M_{\mathcal{X}_G} \times E$. Let D_V be the (effective) divisor in $M_{\mathcal{X}_G} \times E$ formed by irreducible components of this subscheme with multiplicities naturally given by the scheme structure. We denote by $\mathcal{O}(-D_V)$ the invertible sheaf on $M_{\mathcal{X}_G} \times E$ whose sections are functions vanishing along D_V (with the multiplicities prescribed by the divisor). Then

$$\Theta(V) = j^*\mathcal{O}(-D_V), \quad \text{where} \quad j : M_{\mathcal{X}_G} \to M_{\mathcal{X}_G} \times E \quad \text{takes} \quad x \mapsto (x, 0). \tag{2.1.1}$$

It follows that

$$\Theta(V \oplus V') = \Theta(V) \otimes \Theta(V'), \quad \Theta(V^*) = \Theta(V). \tag{2.1.2}$$

The first of the above equalities allows us to extend Θ to a group homomorphism $K_G(M) \to \mathrm{Pic}(M_{\mathcal{X}_G})$, i.e. to take V to be a virtual vector bundle. The product in cohomology (see (1.4.3)) and the contravariant functoriality with respect to the projection $\mathrm{Th}(V) \to M$, see (1.4.1), gives a sheaves morphism on $\mathcal{X}_G$

$$\text{(2.1.3)} \qquad \mathrm{Ell}^i_G(M) \otimes \pi_{M*}\Theta(V) \to \mathrm{Ell}^i_G(\mathrm{Th}(V)).$$

One proves, glueing along the G-cells of M, that (2.1.3) is an isomorphism. It is the (twisted) "Thom isomorphism".

In particular, when V is trivial as a vector bundle, i.e. comes from a representation V of G, the Thom space $\mathrm{Th}(V)$ is the "V-suspension" $\Sigma^V(M)$ in the sense of tom Dieck [Di] (see also [LMS]). So (2.1.3) replaces the datum (c) from Section 1 of [Di].

(2.1.4) Examples.

(2.1.4.1) Let $G = U(n)$ and M be the G-manifold of complete flags in $\mathbf{C}^n$ with standard action. Given a non-trivial character θ of the torus $H = U(1)^n$ let $L_\theta = G \times_H \mathbf{C}_\theta$ be the corresponding line bundle on M and $\mathcal{X}_\theta : E^n \to E$ be the homomorphism of algebraic groups given by the exponents entering into θ. Then (see (1.10.3))

$$\Theta(L_\theta) = \mathcal{O}_{E^n}(-\mathrm{Ker}\,\mathcal{X}_\theta) \in \mathrm{Pic}\,(E^n).$$

Let $M \overset{i}{\hookrightarrow} \mathrm{Th}(L_\theta) \xrightarrow{\pi} M$ be respectively the inclusion of the zero section and the projection. The pull-back morphism $i^* : \mathcal{O}_{E^n}(-\mathrm{Ker}\,\mathcal{X}_\theta) \to \mathcal{O}_{E^n}$ can be seen as a holomorphic section of $\mathcal{O}_{E^n}(\mathrm{Ker}\,\mathcal{X}_\theta)$. Since i^* admits a right inverse, π^*, the divisor of this section is precisely $\mathrm{Ker}\,\mathcal{X}_\theta$.

(2.1.4.2) Let G a compact Lie group and M a connected manifold with trivial G-action. Thus,

$$\pi_M : M_{\mathcal{X}_G} = M_S \times \mathcal{X}_G \to \mathcal{X}_G$$

is the second projection (here $S = \mathcal{X}_{\{1\}}$ is the modular curve). Let V be a G-equivariant complex vector bundle of rank r on M. Since the character group of G is discrete, the action of G on the fibers of V is constant. Denote $\mathbf{C}^r_\theta$ the representation of G on a fiber of V, and $\theta : G \to U(r)$ the corresponding morphism of groups. Let $\Theta(\mathbf{C}^r_\theta) \in \mathrm{Pic}(\mathcal{X}_G)$ be the Thom bundle. Then, $\Theta(V) = \pi^*_M \Theta(\mathbf{C}^r_\theta)$.

(2.2) Tubular neighborhood and elliptic homology. Let (M, N) be a finite G-pair. We denote by $\Theta^i_N(M)$ the coherent sheaf on $N_{\mathcal{X}_G}$ such that $\pi_{N*}\Theta^i_N(M) = \mathrm{Ell}^i_G(M, M \setminus N)$. (We can replace N by a small G-stable tubular neighborhood T and consider $\mathrm{Ell}^i_G(M, M \setminus \mathrm{Int}(T))$, so as not to leave the category of CW-pairs).

In the particular case when $N \hookrightarrow M$ is an embedding of complex manifolds with G acting by holomorphic transformations, $\Theta^{2i}_N(M) = \Theta(T_N M) \otimes \omega^{\otimes(-i)}$ is the twisted Thom sheaf of the normal bundle of N in M, and $\Theta^{2i+1}_N(M) = 0$. In general, the graded sheaf $\Theta_N(M) = \bigoplus_i \Theta^i_N(M)$ is the elliptic analog of the Borel-Moore homology of N. It depends, however, on the ambient space M. In particular, if M is a G-manifold and $F \subset X \subset M$ are closed G-subcomplexes of M, the analogue of the long exact sequence in homology associated to X, F and $U = X \setminus F$ is the following exact sequence of sheaves on $\mathcal{X}_G$ (see (1.7.6)) :

$$\text{(2.2.1)} \qquad \ldots \to \pi_{F*}\Theta^i_F(M) \to \pi_{X*}\Theta^i_X(M) \to \pi_{U*}\Theta^i_U(M \setminus F) \xrightarrow{\partial} \pi_{F*}\Theta^{i+1}_F(M) \to \ldots$$

We have an obvious map of sheaves on $\mathcal{X}_G$

$$\pi_{N*}\Theta_N(M) = \mathrm{Ell}_G(M, M \setminus N) \to \mathrm{Ell}_G(M)$$

which (for smooth N) can be seen as a particular instance of direct functoriality (Gysin map) for elliptic cohomology. Let us consider this in general.

(2.3) Gysin maps. Let $\phi : M \to N$ be a holomorphic map of complex G-manifolds, and $d = \dim M - \dim N$ be the relative dimension of ϕ. The contravariant functoriality with respect to ϕ gives a morphism of schemes $\phi_{\mathcal{X}_G} : M_{\mathcal{X}_G} \to N_{\mathcal{X}_G}$ (see (1.7.4)). As such, it consists of a morphism of topological spaces (still denoted $\phi_{\mathcal{X}_G}$) and a morphism $\phi^{-1}_{\mathcal{X}_G}\mathcal{O}_{N_{\mathcal{X}_G}} \to \mathcal{O}_{M_{\mathcal{X}_G}}$ of sheaves on $M_{\mathcal{X}_G}$. Let us denote by $\Theta(\phi)$ the Thom sheaf on $M_{\mathcal{X}_G}$

corresponding to the virtual vector bundle $\phi^*TN - TM$ on M. If ϕ is proper, the direct image (or Gysin map) is a morphism

$$\phi_* : \Theta(\phi) \to \phi_{\mathcal{X}_G}^{-1}\mathcal{O}_{N_{\mathcal{X}_G}} \tag{2.3.1}$$

of sheaves on $M_{\mathcal{X}_G}$. The construction is achieved in a standard way, by factoring ϕ through G-equivariant complex oriented embedding into a Euclidean space and then considering the tubular neighborhoods [LMS]. Equivalently, we can view ϕ_* as a morphism

$$\phi_* : \phi_{\mathcal{X}_G\,*}\Theta(\phi) \to \mathcal{O}_{N_{\mathcal{X}_G}} \tag{2.3.2}$$

of sheaves on $N_{\mathcal{X}_G}$. This last map of sheaves is a homomorphism of $\mathcal{O}_{N_{\mathcal{X}_G}}$ -modules. This condition is the analog of the projection formula for the ordinary Gysin morphism. Taking the projection to $\mathcal{X}_G$, one gets the following morphism of sheaves on $\mathcal{X}_G$

$$\pi_{M\,*}\Theta(\phi) \to \pi_{N\,*}\mathcal{O}_{N_{\mathcal{X}_G}} = \mathrm{Ell}_G(N) \tag{2.3.3}$$

(2.4) The direct image of a composition. The formula $(\psi\phi)_* = \psi_*\phi_*$ for direct images in traditional theory is modified in our sheaf theoretic approach as follows. Let $M \xrightarrow{\phi} N \xrightarrow{\psi} L$ be two morphisms of complex G-manifolds. Then

$$\Theta(\psi\phi) = \Theta(\phi) \otimes \phi_{\mathcal{X}_G}^*\Theta(\psi), \tag{2.4.1}$$

and we have a natural commutative diagram :

$$\begin{array}{ccccc} & \pi_{M\,*}(\Theta(\phi) \otimes \phi_{\mathcal{X}_G}^*\Theta(\psi)) & \xrightarrow{=} & \pi_{M\,*}\Theta(\psi\phi) & \\ \phi_* \otimes \mathrm{Id} & \downarrow & & \downarrow & (\psi\phi)_* \\ & \pi_{N\,*}\Theta(\psi) & \xrightarrow{\psi_*} & \mathrm{Ell}_G(L) & \end{array} \tag{2.4.2}$$

A little bit more transparent view of direct images and their composition properties can be obtained by a certain twisting described in the following

(2.4.3) Proposition. *For every commutative diagram of proper morphisms of complex G-manifolds*

$$\begin{array}{ccccc} & M & \xrightarrow{\phi} & N & \\ \alpha & \searrow & & \swarrow & \beta \\ & & Y & & \end{array}$$

the direct image map ϕ_* gives rise to a morphism $\phi_* : \pi_{M\,*}\Theta(\alpha) \longrightarrow \pi_{N\,*}\Theta(\beta)$. The correspondence $M \mapsto \pi_{M\,*}\Theta(\alpha)$ extends to a covariant functor from the category of complex G-manifolds over Y (in which morphisms are commutative triangles) to the category of sheaves on $\mathcal{X}_G$.

(2.4.4) Remark. Let $f : M \to N$ be a map between to G-manifolds M and N and let V be a G-vector bundle on N. Denote by $i : N \hookrightarrow V$ the zero section. By applying (2.4.1) to $M \xrightarrow{f} N \xrightarrow{i} V$ we get

$$f^*\Theta(V) = f^*\Theta(i) = \Theta(i\,f) \otimes \Theta(f)^{-1}.$$

Then, a simple computation gives : $f^*\Theta(V) = \Theta(f^*V)$. In other words, Θ is a morphism of contravariant functors.

(2.5) The singular case. We define Thom sheaf and Gysin map for any proper map possibly singular complex algebraic G-varieties by including ψ into a commutative diagram

$$\begin{array}{cccc} & S & \xrightarrow{\psi} & T & \\ j & \downarrow & & \downarrow & k \\ & M & \xrightarrow{\phi} & N & \end{array} \tag{2.5.1}$$

where j, k are closed embeddings, M, N are smooth. Then we define

$$\Theta(\psi) = \Theta(j) \otimes \mathcal{H}om\big(\psi^*_{\mathcal{X}_G}\Theta(k), j^*_{\mathcal{X}_G}\Theta(\phi)\big). \tag{2.5.2}$$

It is easy to verify that such a definition of $\Theta(\psi)$ is indeed independent of the choice of j, ϕ and k. All the statements of (2.2-4) extend to this case without any further modifications. Note that with this definition, for any diagram of proper morphisms of form (2.5.1) we will have equality (2.5.2).

(2.6) Euler class. Let M be a finite G-complex and V be a complex G-vector bundle over M. Let $i : M \hookrightarrow V$ be the zero section. The morphism of schemes $i_{\mathcal{X}_G} : M_{\mathcal{X}_G} \to V_{\mathcal{X}_G}$ is an isomorphism by the homotopy invariance of elliptic cohomology. The map i is proper. The Gysin map $i_* : \Theta(V) \to i^{-1}_{\mathcal{X}_G}\mathcal{O}_{V_{\mathcal{X}_G}} = \mathcal{O}_{M_{\mathcal{X}_G}}$ can be seen as an regular section of the line bundle $\Theta(V)^{-1}$ on $M_{\mathcal{X}_G}$. Denote this section by $e_V \in H^0(M_{\mathcal{X}_G}, \Theta(V)^{-1})$. In our situation, e_V is the analogue of the usual Euler class in equivariant cohomology or K-theory. In particular, if V and W are complex vector bundles on M one has $e_{V\oplus W} = e_V \otimes e_W$ as a section of $\Theta(V \oplus W)^{-1} = \Theta(V)^{-1} \otimes \Theta(W)^{-1}$.

(2.6.1) Example. Let $\theta : G \to U(1)$ be a character of G. Take $M = pt$ and $V = \mathbf{C}_\theta$ the corresponding one-dimensional representation of G. The characteristic class of $\mathbf{C}_\theta$ is the group homomorphism $\mathcal{X}_\theta : \mathcal{X}_G \to \mathcal{X}_{U(1)} = E$. Thus, $\Theta(\mathbf{C}_\theta) = \mathcal{O}(-\mathrm{Ker}\,\mathcal{X}_\theta) \in \mathrm{Pic}(\mathcal{X}_G)$. The same argument as in (2.1.4.1) shows that the Euler class $e_{\mathbf{C}_\theta}$ is a regular section of $\mathcal{O}(\mathrm{Ker}\,\mathcal{X}_\theta)$ over $\mathcal{X}_G$ with zeros precisely on $\mathrm{Ker}\,\mathcal{X}_\theta$. Note that $\mathrm{Ker}\,\mathcal{X}_\theta = \mathcal{X}_{\mathrm{Ker}\,\theta}$ (see (1.4.4)).

(2.7) Action of correspondences on elliptic cohomology. By a correspondence between smooth complex algebraic G-varieties M_1 and M_2 we mean a G-stable (possibly singular) subvariety $W \subset M_1 \times M_2$ such that the projection $q_2 : W \to M_2$ is proper. Any correspondence defines a morphism of sheaves on $\mathcal{X}_G$

$$\pi_{W\,*}\Theta(q_2) \longrightarrow \mathcal{H}om_{\mathcal{X}_G}(\mathrm{Ell}_G(M_1), \mathrm{Ell}_G(M_2)), \tag{2.7.1}$$

which is defined as follows. Given any (local) section s of $\Theta(q_2)$ on $W_{\mathcal{X}_G}$ and a function g on $M_{1\mathcal{X}_G}$ (i.e., a section of $\mathrm{Ell}_G(M_1)$), we first lift g to a function on $W_{\mathcal{X}_G}$ by means of the scheme map $q_{1\,\mathcal{X}_G} : W_{\mathcal{X}_G} \to M_{1\mathcal{X}_G}$ associated to the first projection $q_1 : W \to M_1$. Multiplying by s, yields another section of $\Theta(q_2)$, and applying q_{2*} we get a function on $M_{1\mathcal{X}_G}$.

Given two correspondences $W_{12} \subset M_1 \times M_2$ and $W_{23} \subset M_2 \times M_3$, define their composition as the correspondence

$$W_{12} \circ W_{23} = \big\{(m_1, m_3) \in M_1 \times M_3 \mid \exists m_2 \in M_2 \text{ s.t. } (m_1, m_2) \in W_{12},\ (m_2, m_3) \in W_{23}\big\}.$$

Let $q_2 : W_{12} \to M_2$, $q_3 : W_{23} \to M_3$ and $\overline{q}_3 : W_{13} = W_{12} \circ W_{23} \to M_3$ be the second projections. The composition of actions of correspondences on Ell_G lifts to a morphism of sheaves on $\mathcal{X}_G$

$$\pi_{W_{12}\,*}\Theta(q_2) \otimes \pi_{W_{23}\,*}\Theta(q_3) \to \pi_{W_{13}\,*}\Theta(\overline{q}_3) \tag{2.7.2}$$

which is defined as follows. Consider first $W_{12} \times_{M_2} W_{23}$, the fiber product of W_{23} and W_{12} over M_2. Then we have the diagram

$$\begin{tikzcd}
W_{12} \times_{M_2} W_{23} \arrow[r, "q_{23}"] \arrow[d, "q_{12}"'] & W_{23} \arrow[r, "q_3"] \arrow[d] & M_3 \\
W_{12} \arrow[r, "q_2"] \arrow[d, "q_1"'] & M_2 & \\
M_1 & &
\end{tikzcd}$$

The diagram yields

$$\Theta(q_3 q_{23}) = q_{12}^*\Theta(q_2) \otimes q_{23}^*\Theta(q_3),$$

and thus we have a multiplication

$$\pi_{W_{12}}{}_*\Theta(q_2) \otimes \pi_{W_{23}}{}_*\Theta(q_3) \to \pi_{W_{12} \times_{M_2} W_{23}}{}_*\Theta(q_3 q_{23}).$$

The map (2.7.2) is obtained by composing this multiplication with the map

$$\pi_{W_{12} \times_{M_2} W_{23}}{}_*\Theta(q_3 q_{23}) \to \pi_{W_{13}}{}_*\Theta(\overline{q}_3)$$

induced, as in (2.4.3), by the natural projection $W_{12} \times_{M_2} W_{23} \to W_{13}$. The maps (2.7.2) are associative in an obvious sense. In particular, if $M_1 = M_2 = M_3 = M$ we get

(2.7.3) Proposition. *If $W \subset M \times M$ is a correspondence such that $W \circ W = W$, then $\pi_{W}{}_*\Theta(q_2)$ has a natural structure of a sheaf of associative algebras on $\mathcal{X}_G$.*

(2.7.4) Remark. The morphism (2.7.1) is nothing but the particular case of (2.7.2) corresponding to $M_1 = pt$.

(2.8) Lagrangian correspondences and their action on Ell_G**.** We will apply a slight modification of the formalism above to the cotangent bundles. Given a complex manifold M, let TM (resp. T^*M) denote its tangent (resp. cotangent) bundle. Write $T_W M$ and $T^*_W M$ for the normal and conormal bundles of a submanifold $W \subset M$. It is well known that T^*M is a symplectic manifold, and $T^*_W M$ is a Lagrangian submanifold in T^*M.

Let M_1, M_2 be two smooth complex algebraic G-varieties $N_1 = T^*M_1$, $N_2 = T^*M_2$. By a Lagrangian correspondence between N_1 and N_2 we mean a G-stable possibly singular Lagrangian subvariety $Z \subset T^*(M_1 \times M_2)$ whose projections to T^*M_1, T^*M_2 are proper. In this case we have the diagram

$$M_1 \xleftarrow{\pi_1} T^*M_1 \xleftarrow{p_1} Z \xrightarrow{p_2} T^*M_2 \xleftarrow{i_2} M_2, \tag{2.8.1}$$

where π_1 is the projection of the cotangent bundle, and i_2 is the zero section. To a Lagrangian correspondence Z we associate the coherent sheaf

$$\begin{aligned} \Xi_Z &= \Theta(p_2) \otimes \Theta(p_1^*\pi_1^* T^*M_1) \otimes \Theta(p_2^*\pi_2^* T^*M_2)^{-1} = \\ &= \Theta_Z(T^*(M_1 \times M_2)) \otimes \Theta(p_1^*\pi_1^* T^*M_1)^{-1} \otimes \Theta(p_2^*\pi_2^* T^*M_2)^{-1} \end{aligned} \tag{2.8.2}$$

on $Z_{\mathcal{X}_G}$ called the *microlocal Thom sheaf* of Z. There is a map

$$\pi_{Z}{}_*\Xi_Z \to \mathcal{H}om_{\mathcal{X}_G}(\pi_{M_1}{}_*\Theta(T^*M_1)^{-1}, \pi_{M_2}{}_*\Theta(T^*M_2)^{-1}) \tag{2.8.3}$$

defined as follows. Suppose we have a local section of Ξ_Z which we take in the form $a \otimes b \otimes c$ according to the factorization in the first line of (2.8.2). Given a section s of $\Theta(T^*M_1)^{-1}$, the product sb is a function on $M_{1\mathcal{X}_G}$, we lift it to $Z_{\mathcal{X}_G}$ by $\pi_{1\mathcal{X}_G}$ and $p_{1\mathcal{X}_G}$, then multiply by a, getting a section of $\Theta(p_2)$, project along p_2, getting a function on $(T^*M_2)_{\mathcal{X}_G}$, pull it back to $M_{2\mathcal{X}_G}$ by $i_{2\mathcal{X}_G} : M_{2\mathcal{X}_G} \xrightarrow{\sim} (T^*M_2)_{\mathcal{X}_G}$, and multiply it with c, getting a section of $\Theta(T^*M_2)^{-1}$.

As for the non-modified case (2.7.2), given two Lagrangian correspondences $Z_{12} \subset N_1 \times N_2$, $Z_{23} \subset N_2 \times N_3$, and their composition $Z_{13} = Z_{12} \circ Z_{23}$, the composition of actions of correspondences on Thom bundles as in (2.8.3) lifts to an associative morphism of sheaves

$$\pi_{Z_{12}}{}_*\Xi_{Z_{12}} \otimes \pi_{Z_{23}}{}_*\Xi_{Z_{23}} \to \pi_{Z_{13}}{}_*\Xi_{Z_{13}}. \tag{2.8.4}$$

In particular, if $M_1 = M_2 = M$ we get the following :

(2.8.5) Proposition. *If $Z \subset T^*M \times T^*M$ is a Lagrangian correspondence such that $Z \circ Z = Z$, then $\pi_{Z}{}_*\Xi_Z$ has a natural structure of a sheaf of associative algebras on $\mathcal{X}_G$, and $\pi_{M}{}_*\Theta(T^*M)^{-1}$ is a sheaf of its modules.*

A natural example of a Lagrangian correspondence is obtained as follows. Let $W \subset M_1 \times M_2$ be a smooth closed submanifold whose projections q_1 and q_2 to M_1 and M_2 are proper. Take $Z = T^*_W(M_1 \times M_2)$. Then Z is a Lagrangian correspondence between N_1 and N_2 while W is a correspondence between M_1 and M_2. We have

(2.8.6) Proposition. *Under the above assumptions, the sheaf Ξ_{Z} is naturally identified with $\mathcal{O}_{Z\mathcal{X}_G}$.*

Proof. The zero section map $i : W \to Z = T^*_W(M_1 \times M_2)$ is a G-homotopy equivalence, so $i_{\mathcal{X}_G} : W\mathcal{X}_G \to Z\mathcal{X}_G$ is an isomorphism of schemes. Thus it suffices to proove that $i^*_{\mathcal{X}_G}\Theta(p_2) \simeq \mathcal{O}_{W\mathcal{X}_G}$. Recall that Θ factors through $K_G(W)$, the Grothendieck group of G-vector bundles on W. Denoting by $[V]$ the class in $K_G(M)$ of a bundle V, we find

$$q_1^*[T^*M_1] = [T^*q_2] + [T^*_W(M_1 \times M_2)].$$

Since Θ is insensitive to the dualization (see (2.1.2)) we get

$$\Theta(q_2) \otimes \Theta(q_1^*T^*M_1) \otimes \Theta(i)^{-1} = \mathcal{O}_{W\mathcal{X}_G}. \tag{2.8.7}$$

In another hand, formula (2.4.1) applied to the commutative square

$$\begin{array}{ccccc} & Z & \xrightarrow{p_2} & N_2 & \\ i & \uparrow & & \uparrow & i_2 \\ & W & \xrightarrow{q_2} & M_2 & \end{array}$$

gives $\Theta(q_2) \otimes {q_2}^*_{\mathcal{X}_G}\Theta(i_2) = \Theta(i) \otimes i^*_{\mathcal{X}_G}\Theta(p_2)$. Combining it with (2.8.2) and (2.8.7) we get $i^*_{\mathcal{X}_G}\Xi_{Z} = \mathcal{O}_{W\mathcal{X}_G}$.

(2.8.8) Remarks.
(2.8.8.1) If $W \subset M_1 \times M_2$ is a smooth locally closed submanifold and $Z = T^*_W(M_1 \times M_2)$ set $\partial Z = \overline{Z} \setminus Z$ and define

$$\Xi_Z = \Theta_{Z}(N_1 \times N_2 \setminus \partial Z) \otimes \Theta(p_1^*\pi_1^*T^*M_1)^{-1} \otimes \Theta(p_2^*\pi_2^*T^*M_2)^{-1}.$$

Then we still have $\Xi_Z = \mathcal{O}_{Z\mathcal{X}_G}$.
(2.8.8.2) Let Z' be a G-stable Lagrangian subvariety of Z. Let $i : Z' \to Z$ and $p_2' : Z' \to M_2$ be the inclusion in Z and the projection to M_2. Formula (2.4.1) gives

$$\Xi_{Z'} = \Theta(i) \otimes i^*_{\mathcal{X}_G}\Xi_{Z}.$$

Thus, the Gysin map induces a morphism ${i_{\mathcal{X}_G}}_*\Xi_{Z'} \to \Xi_{Z}$ which commutes with the action of ${\pi_{Z}}_*\Xi_Z$ and ${\pi_{Z'}}_*\Xi'_{Z}$ on elliptic cohomology, as described in (2.8.3).
(2.8.8.3) The morphism (2.8.3) is nothing but the particular case of (2.8.4) corresponding to $M_1 = N_1 = pt$.

(2.9) Localization. Suppose that G is a compact Abelian Lie group and M a complex smooth manifold. For any $g \in G$ put $\mathcal{X}_g = \mathcal{X}_G \setminus \bigcup_{g \notin H} \mathcal{X}_H$, the union beeing taken over all the closed subgroups H of G not containing g; the set $\mathcal{X}_g$ is non-empty (see example (1.4.4)). The fixed-points set M^g is a smooth complex subvariety of M. Let $i : M^g \hookrightarrow M$ be the inclusion map; it is a proper map.

(2.9.1) Proposition. *For any $g \in G$, the Gysin map $i_* : {\pi_{M^g}}_*\Theta(T_{M^g}M) \to \mathrm{Ell}_G(M)$ is an isomorphism over $\mathcal{X}_g \subset \mathcal{X}_G$. Besides, the morphism of sheaves $i^*i_* : {\pi_{M^g}}_*\Theta(T_{M^g}M) \to \mathrm{Ell}_G(M^g)$ is the multiplication by the Euler class $e_{T_{M^g}M}$ and is invertible over $\mathcal{X}_g \subset \mathcal{X}_G$.*

Proof. Denote by $j : M \setminus M^g \to M$ the inclusion. The long exact sequence (2.2.1) gives :

$$... \xrightarrow{\partial} {\pi_{M^g}}_*\Theta^i(T_{M^g}M) \xrightarrow{i_*} \mathrm{Ell}^i_G(M) \xrightarrow{j^*} \mathrm{Ell}^i_G(M \setminus M^g) \xrightarrow{\partial} {\pi_{M^g}}_*\Theta^{i+1}(T_{M^g}M) \to ...$$

So it suffices to prove that, as a sheaf over $\mathcal{X}_G$, $\mathrm{Ell}^0_G(M \setminus M^g)$ is supported on $\mathcal{X}_G \setminus \mathcal{X}_g$. By a standard argument we are reduced to the case where $M \setminus M^g$ is a G-orbit, say G/H, with $g \notin H$. Then, we know from the induction axiom (1.6.2) that $\mathrm{Ell}^0_G(G/H) = \mathcal{O}_{\mathcal{X}_H}$. Thus, the first part of the proposition follows from $\mathcal{X}_H \cap \mathcal{X}_g = \emptyset$. The second part of the proposition is a direct corollary of the construction of Euler classes (see (2.6)), except for the invertiblity over $\mathcal{X}_g$ which follows from standard arguments and (2.6.1).

Let $f : M_1 \to M_2$ be a proper morphism of complex smooth varieties. Proposition (2.4.3) applied to the diagram

$$\begin{array}{ccccc} & M_1^g & \stackrel{i_1}{\hookrightarrow} & M_1 & \\ f^g & \downarrow & & \downarrow & f\,, \\ & M_2^g & \stackrel{i_2}{\hookrightarrow} & M_2 & \end{array}$$

where i_1, i_2 are the inclusions and f^g is the restriction of f to M_1^g and M_2^g, gives the following commutative square

$$\textbf{(2.9.2)}\qquad \begin{array}{ccc} \pi_{M_1^g\,*}\Theta(TM_1^g)^{-1} & \stackrel{i_{1*}}{\longrightarrow} & \pi_{M_1\,*}\Theta(TM_1)^{-1} \\ f_*^g \downarrow & & \downarrow f_* \\ \pi_{M_2^g\,*}\Theta(TM_2^g)^{-1} & \stackrel{i_{2*}}{\longrightarrow} & \pi_{M_2\,*}\Theta(TM_2)^{-1} \end{array}$$

where i_{1*} and i_{2*} are given by multiplication by $e_{T_{M_1^g}M_1}$ and $e_{T_{M_2^g}M_2}$. The particular case of the projection $p\,:\,M\to pt$ gives the Lefschetz-type formula :

$$\textbf{(2.9.3)}\qquad \forall s\in\pi_{M\,*}\Theta(p),\quad p_*(s)=p_*^g(s\cdot e_{T_{M^g}M}^{-1})$$

where $s\cdot e_{T_{M^g}M}^{-1}$ is a meromorphic section of $\pi_{M^g\,*}\Theta(p^g)$ without poles on $\mathcal{X}_g\subset\mathcal{X}_G$.

(2.10) Localization of correspondences. Suppose that G is a compact Abelian Lie group. Let M_1, M_2, two smooth complex algebraic G-varieties and W a correspondence between M_1 and M_2, i.e. a G-stable (possibly singular) subvariety $W\subset M_1\times M_2$ such that the second projection, q_2, is proper. From formula (2.4.1) we get

$$\Theta(q_2)=\Theta_W(M_1\times M_2)\otimes\Theta(q_1^*TM_1)^{-1}.$$

For any $g\in G$, denote by i the inclusion $W^g\hookrightarrow W$. The contravariant functoriallity of Ell_G with respect to the inclusion of pairs $(M_1^g\times M_2^g,\,M_1^g\times M_2^g\setminus W^g)\subset(M_1\times M_2,\,M_1\times M_2\setminus W)$ gives a morphism of sheaves over $W_{\mathcal{X}_G}$

$$i^*\,:\,\Theta_W(M_1\times M_2)\to i_{\mathcal{X}_G\,*}\Theta_{W^g}(M_1^g\times M_2^g).$$

It induces a morphism of sheaves

$$\Theta(q_2)\to i_{\mathcal{X}_G\,*}\Theta_{W^g}(M_1^g\times M_2^g)\otimes\Theta(q_1^*TM_1)^{-1}=i_{\mathcal{X}_G\,*}\big(\Theta(q_2^g)\otimes\Theta(q_1^{g*}T_{M_1^g}M_1)^{-1}\big).$$

The Euler class $e_{T_{M_1^g}M_1}$ is a regular section of $\Theta(T_{M_1^g}M_1)^{-1}$. Thus, multiplying by $e_{T_{M_1^g}M_1}^{-1}$ gives a rational morphism of sheaves $r_g\,:\,\pi_{W\,*}\Theta(q_2)\to\pi_{W^g\,*}\Theta(q_2^g)$.

(2.10.1) Proposition. *For any $g\in G$, the map $r_g\,:\,\pi_{W\,*}\Theta(q_2)\to\pi_{W^g\,*}\Theta(q_2^g)$ is regular and invertible over $\mathcal{X}_g$. Moreover r_g commutes with the product of correspondences, as defined in (2.7.2).*

Proof. The first part of the proposition follows from (2.9). With the notations of (2.7.2), the second part is reduced to the commutativity of the following diagram

$$\textbf{(2.10.2)}\qquad \begin{array}{ccc} \pi_{W_{12}\,*}\Theta(q_2)\otimes\pi_{W_{23}\,*}\Theta(q_3) & \stackrel{r_g\otimes r_g}{\longrightarrow} & \pi_{W_{12}^g\,*}\Theta(q_2^g)\otimes\pi_{W_{23}^g\,*}\Theta(q_3^g) \\ \downarrow & & \downarrow \\ \pi_{W_{13}\,*}\Theta(\overline{q}_3) & \stackrel{r_g}{\longrightarrow} & \pi_{W_{13}^g\,*}\Theta(\overline{q}_3^g) \end{array},$$

whose proof is done exactly as in equivariant K-theory (see [CG; Theorem 4.10.12]).

(2.10.3) Similarly, if $N_1=T^*M_1$, $N_2=T^*M_2$, and $Z\subset N_1\times N_2$ is a Lagrangian correspondence, the Proposition (2.10.1) has the following analogue, with the notations of (2.8) : contravariant functoriality with respect to the inclusion, i, of fixed points subvarieties gives a map

$$i^*\,:\,\Theta_Z(N_1\times N_2)\to i_{\mathcal{X}_G\,*}\Theta_{Z^g}(N_1^g\times N_2^g).$$

By composing it with the product by $e^{-1}_{T_{M_1^g}M_1} \otimes e^{-1}_{T_{M_2^g}M_2}$, we get a rational morphism of sheaves $\rho_g : \pi_{Z*}\Xi_Z \to \pi_{Z^g*}\Xi_{Z^g}$, which is regular and invertible over $\mathcal{X}_g$ and commutes with the product of correspondences.

3. Geometric construction of current algebras.

(3.1) Let X be a smooth irreducible complex algebraic variety. For any $d \geq 0$ we have a natural projection

$$\pi : X^d \to X^{(d)} = X^d/S_d,$$

where $X^{(d)}$ is the d-fold symmetric product of X. For any coherent $\mathcal{O}_X$-sheaf $\mathcal{F}$ we construct coherent sheaves $\mathcal{F}^{(\oplus d)}$ and $\mathcal{F}^{(\otimes d)}$ on $X^{(d)}$ as follows :

$$\mathcal{F}^{(\oplus d)} = \left(\pi_*\left(\bigoplus_{i=1}^{d} \mathrm{pr}_i^*\mathcal{F}\right)\right)^{S_d}, \quad \mathcal{F}^{(\otimes d)} = \left(\pi_*\left(\bigotimes_{i=1}^{d} \mathrm{pr}_i^*\mathcal{F}\right)\right)^{S_d},$$

where $\mathrm{pr}_i : X^d \to X$ is the ith projection. Thus the geometric fiber of $\mathcal{F}^{(\oplus d)}$ (resp. $\mathcal{F}^{(\otimes d)}$) at a point $I = \{x_1, ..., x_d\} \in X^{(d)}$ is given by $\bigoplus_{x \in I} \mathcal{F}_x$ (resp. $\bigotimes_{x \in I} \mathcal{F}_x$).

(3.2) Let $\mathcal{G}$ be a locally free $\mathcal{O}_X$-sheaf of Lie algebras (with bracket not necessarily $\mathcal{O}_X$-linear). Then $\mathcal{G}^{(\oplus d)}$ has an obvious structure of a sheaf of Lie algebras on $X^{(d)}$. Given a locally free $\mathcal{O}_X$-sheaf $\mathcal{M}$ of $\mathcal{G}$- modules, we find, in the same way, that $\mathcal{M}^{(\otimes d)}$ is a sheaf of $\mathcal{G}^{(\oplus d)}$-modules so that there is a natural Lie algebra morphism

$$\mathcal{G}^{(\oplus d)} \to \mathcal{E}nd\mathcal{M}^{(\otimes d)} \qquad (\mathbf{3.2.1})$$

For instance, if X is a point, then $\mathcal{G}$ is a finite-dimensional Lie algebra, $\mathcal{M}$ is a finite-dimensional $\mathcal{G}$-module, $\mathcal{G}^{(\oplus d)} = \mathcal{G}$, and $\mathcal{M}^{(\otimes d)} = S^d\mathcal{M}$.

Observe that for any open set $U \subset X$ we have the "diagonal" homomorphism of Lie algebras $\Gamma(U, \mathcal{G}) \to \Gamma(U^{(d)}, \mathcal{G}^{(\oplus d)})$. Hence the space $\Gamma(U^{(d)}, \mathcal{M}^{(\otimes d)})$ acquires a natural structure of $\Gamma(U, \mathcal{G})$- module.

We will be mainly concerned with the case $\mathcal{G} = \mathcal{E}nd(\mathcal{M})$, where $\mathcal{M}$ is (the sheaf of sections of) a vector bundle on X. For any open $U \in X$ the above construction makes $\Gamma(U^{(d)}, \mathcal{M}^{(\oplus d)})$ into a representation of the "current algebra" $\Gamma(U, \mathcal{E}nd(\mathcal{M}))$.

(3.3) We recall the general formalism of sheaf-theoretic correspondences. Let $s : V \to S$ be a finite morphism of normal algebraic varieties. Form the Cartesian product $M = V \times_S V$ and let $\tilde{M}$ be the normalization of M. We have the natural commutative diagram of projections

$$\begin{array}{ccccc} & \tilde{M} & \xrightarrow{q_2} & V & \\ q_1 & \downarrow & \overset{r}{\searrow} & \downarrow & s\,. \\ & V & \xrightarrow{s} & S & \end{array}$$

We view $\tilde{M}$ as a correspondence from V to V. Let $\mathcal{V}$ be a vector bundle on V. We define the following sheaves on S :

$$\mathcal{W} = s_*\mathcal{V}, \quad \tilde{\mathcal{G}} = r_*\mathcal{H}om(q_2^*\mathcal{V}, q_1^*\mathcal{V}).$$

The composition of $\mathcal{H}om$ makes $\tilde{\mathcal{G}}$ into a sheaf of associative algebras on S, and $\mathcal{W}$ is a sheaf of $\tilde{\mathcal{G}}$-modules.

(3.3.1) Remark. The structure of $\tilde{\mathcal{G}}$-module on $\mathcal{W}$ can be described in terms similar to those in (2.7.1). Indeed, fix s and g two local sections of $\tilde{\mathcal{G}}$ and $\mathcal{W}$. They can be seen as sections of $\mathcal{H}om(q_2^*\mathcal{V}, q_1^*\mathcal{V})$ and $\mathcal{V}$. Lift g to a section of $q_1^*\mathcal{V}$ and apply s to it, we get a section $s(g)$ of $q_2^*\mathcal{V}$. Now, since q_2 is flat and finite and

$\mathcal{V}$ is locally trivial we have a *trace* map $q_{2*}q_2^*\mathcal{V} \to \mathcal{V}$; just take the image of $s(g)$ by this trace. This way we obtain a Lie algebra map

$$\tilde{\mathcal{G}} \to \mathcal{E}nd(\mathcal{W}) \tag{3.3.2}$$

which is clearly injective by construction.

(3.4) Let now $q : Y \to X$ be an unramified covering of a smooth irreducible variety X (Y may be disconnected), $\mathcal{L}$ a line bundle on Y and $\mathcal{M} = q_*\mathcal{L}$. In the setup of the previous section, we put $S = X^{(d)}$, $V = Y^{(d)}$, $s = q^{(d)} : Y^{(d)} \to X^{(d)}$ the map induced by q. Put also $\mathcal{V} = \mathcal{L}^{\otimes d}$, $\mathcal{G} = \mathcal{E}nd(\mathcal{M})$, so that $\mathcal{W} = \mathcal{M}^{(\otimes d)}$ and

$$\tilde{\mathcal{G}} = r_*\mathcal{H}om(q_2^*\mathcal{L}^{\otimes d}, q_1^*\mathcal{L}^{\otimes d}) \hookrightarrow \mathcal{E}nd(\mathcal{M}^{\otimes d}).$$

The normalization $\tilde{M}$ of $M = Y^{(d)} \times_{X^{(d)}} Y^{(d)}$ is in this case nothing but $(Y \times_X Y)^{(d)}$, the symmetric power of the smooth variety $Y \times_X Y$. The $\mathcal{G}^{(\oplus d)}$ - action on $\mathcal{M}^{(\otimes d)}$ defined in (3.2), gives a map $\mathcal{G}^{(\oplus d)} \to \mathcal{E}nd(\mathcal{M}^{(\otimes d)})$.

(3.5) Theorem. *There is a natural Lie algebra homomorphism $\tau : \mathcal{G}^{(\oplus d)} \to \tilde{\mathcal{G}}$ of sheaves on $X^{(d)}$ making the following diagram commute:*

$$\begin{tikzcd} \mathcal{G}^{(\oplus d)} \arrow[r, "(3.2.1)"] \arrow[d] & \mathcal{E}nd\,\mathcal{M}^{(\otimes d)} \\ \tilde{\mathcal{G}} \arrow[ur, "(3.3.2)"'] & \end{tikzcd} .$$

Let $\mathcal{U}(L)$ denote the universal enveloping algebra of Lie algebra L. Applying the "diagonal" map to τ, we get the first part of the following proposition :

(3.6) Proposition. *(a) For any open $U \subset X$, we have a Lie algebra homomorphism*

$$\tau_U : \Gamma(U, \mathcal{G}) \to \Gamma(U^{(d)}, \tilde{\mathcal{G}}).$$

(b) The following homomorphism of associative algebras

$$\mathcal{U}(\Gamma(U, \mathcal{G})) \to \Gamma(U^{(d)}, \tilde{\mathcal{G}})$$

induced by τ_U, is surjective.

Proof of Theorem 3.5 will be given in (3.11), the proof of the second part of Proposition 3.6 in (3.14).

(3.7) Partitions and matrices. We need to introduce some combinatorial notation, to be used throughout the paper.

Fix an integer $d \geq 1$. Let $\mathbf{V} \subset \mathbf{N}^n$ be the set of all partitions $\mathbf{v} = (v_1, \ldots, v_n)$ of d into n summands. For $\mathbf{v} \in \mathbf{V}$ let $[\mathbf{v}]_i \subset \{1, ..., d\}$ be the i-th segment of the partition $\mathbf{v}$, i.e., the segment $[v_1{+}...{+}v_{i-1}{+}1,\ v_1{+}...{+}v_i]$. Let $S_{\mathbf{v}} = S_{[\mathbf{v}]_1} \times ... \times S_{[\mathbf{v}]_n}$ be the Young subgroup in the symmetric group S_d, corresponding to $\mathbf{v}$, i.e., the subgroup of permutations preserving each segment $[\mathbf{v}]_i$.

Let $\mathbf{M}$ be the set of the $(n \times n)$-matrices $A = (a_{ij}) \in \mathbf{Z}_+^{n^2}$ with non-negative integral entries such that $\sum_{i,j} a_{ij} = d$. For $\mathbf{v}^1, \mathbf{v}^2 \in \mathbf{V}$ we denote

$$\mathbf{M}(\mathbf{v}^1, \mathbf{v}^2) = \Big\{ A \in \mathbf{M} : \sum_j a_{ij} = v_i^1,\ \sum_i a_{ij} = v_j^2 \Big\},$$

so that $\mathbf{M} = \coprod_{\mathbf{v}^1, \mathbf{v}^2 \in \mathbf{V}} \mathbf{M}(\mathbf{v}^1, \mathbf{v}^2)$. For any given $\mathbf{v}^1, \mathbf{v}^2$ we have a natural map

$$\mathbf{m} = \mathbf{m}_{\mathbf{v}^1, \mathbf{v}^2} : S_d \longrightarrow \mathbf{M}(\mathbf{v}^1, \mathbf{v}^2)$$

which identifies $\mathbf{M}(\mathbf{v}^1, \mathbf{v}^2)$ with the double coset space $S_{\mathbf{v}^1} \backslash S_d / S_{\mathbf{v}^2}$. Namely, $\mathbf{m}$ associates to a permutation $\sigma \in S_d$ the matrix $\mathbf{m}^\sigma$ with

$$\mathbf{m}_{ij}^\sigma = \mathrm{Card}\,\big\{ \alpha \in [\mathbf{v}^1]_i : \sigma(\alpha) \in [\mathbf{v}^2]_j \big\}.$$

Let us introduce a partial order $\preceq$ on $\mathbf{M}(\mathbf{v}^1, \mathbf{v}^2)$ as follows. For $A = (a_{ij})$ and $B = (b_{ij})$ in $\mathbf{M}(\mathbf{v}^1, \mathbf{v}^2)$ we say that $A \preceq B$, if for any $1 \le i < j \le n$ we have

$$\sum_{r\le i\,;\, s\ge j} a_{rs} \le \sum_{r\le i\,;\, s\ge j} b_{rs}$$

and for any $1 \le j < i \le n$ we have

$$\sum_{r\ge i\,;\, s\le j} a_{rs} \le \sum_{r\ge i\,;\, s\le j} b_{rs}.$$

The map $\mathbf{m} : S_d \to \mathbf{M}(\mathbf{v}^1, \mathbf{v}^2)$ is monotone with respect to the Bruhat order (denoted $\le$) on the symmetric group; in other words, $A \le B \Rightarrow A \preceq B$ for any $A, B \in \mathbf{M}(\mathbf{v}^1, \mathbf{v}^2)$ (see [BLM , Lemma 3.6]). The order $\preceq$ on $\mathbf{M} = \coprod \mathbf{M}(\mathbf{v}^1, \mathbf{v}^2)$ is the disjoint union of the orders on the components $\mathbf{M}(\mathbf{v}^1, \mathbf{v}^2)$, i.e., the elements of different components are set to be incomparable.

(3.8) The case $\mathcal{M} = \mathcal{O}_X^n$. We are now going to consider a special case of the construction of (3.4-6). Let X be a smooth irreducible algebraic variety. Recall that $X^{(m)}$ stands for the m-th symmetric power. For $\mathbf{v} \in \mathbf{V}$ set $X^{(\mathbf{v})} = \prod_i X^{(v_i)}$. Similarly, given $A \in \mathbf{M}(\mathbf{v}^1, \mathbf{v}^2)$ we set $X^{(A)} = \prod_{i,j} X^{(a_{ij})}$. so that one has natural projections

$$X^{(\mathbf{v}^1)} \overset{pr_1}{\leftarrow} X^{(A)} \overset{pr_2}{\rightarrow} X^{(\mathbf{v}^2)} \tag{3.8.0}$$

We will also frequently use the following natural maps

$$\pi_{\mathbf{v}} : X^{(\mathbf{v})} \to X^{(d)}, \quad \pi_A : X^{(A)} \to X^{(d)}, \quad p_{ij} : X^{(A)} \to X^{(a_{ij})}$$

Put $[n] = \{1, 2, ..., n\}$. Let $Y = X \times [n]$ and $q : Y \to X$ be the natural projection. Put $\mathcal{L} = \mathcal{O}_Y$, so that $\mathcal{M} = \mathcal{O}_X^n$. Thus $\mathcal{G} = gl_n(\mathcal{O}_X)$ is the sheaf of maps $X \to gl_n(\mathbf{C})$ and $\tilde{\mathcal{G}} = r_*\mathcal{O}_{\tilde{M}}$.

Observe that $Y^{(d)} = \coprod_{\mathbf{v}\in\mathbf{V}} X^{(\mathbf{v})}$. Hence, we get the canonical direct sum decomposition

$$\mathcal{M}^{(\otimes d)} = q_*^{(d)}\mathcal{O}_{Y^{(d)}} = \bigoplus_{\mathbf{v}} \pi_{\mathbf{v}*}\mathcal{O}_{X^{(\mathbf{v})}}. \tag{3.8.1}$$

Following (3.3), we put

$$M = Y^{(d)} \times_{X^{(d)}} Y^{(d)} = \coprod_{\mathbf{v}^1,\mathbf{v}^2} M_{\mathbf{v}^1,\mathbf{v}^2}, \quad M_{\mathbf{v}^1,\mathbf{v}^2} = X^{(\mathbf{v}^1)} \times_{X^{(d)}} X^{(\mathbf{v}^2)}.$$

The variety $M_{\mathbf{v}^1,\mathbf{v}^2}$ is connected but reducible. The irreducible components of this variety are parametrized by $n \times n$ matrices $A = \|a_{ij}\| \in \mathbf{M}(\mathbf{v}^1, \mathbf{v}^2)$. For such a matrix A the irreducible component M_A is the closure of the set

$$\left\{ \big((I_1, ..., I_n), (J_1, ..., J_n)\big) \in X^{(\mathbf{v}^1)} \times X^{(\mathbf{v}^2)} \,\middle|\, \bigcup_\nu I_\nu = \bigcup_\nu J_\nu,\ |I_\mu \cap J_\nu| = a_{\mu\nu} \right\}. \tag{3.8.2}$$

The normalization of the variety M_A is nothing but $X^{(A)}$, so the normalization of M is

$$\tilde{M} = (Y \times_X Y)^{(d)} = (X \times [n]^2)^{(d)} = \coprod_{A\in\mathbf{M}} X^{(A)}.$$

For $1 \le i, j \le n$ let $E_{ij} \in \mathrm{Mat}_n(\mathbf{Z})$ be the standard matrix unit (1 at the spot (i, j) and 0 elsewhere). For every partition $\mathbf{v} = (v_1, ..., v_n)$ of d and $i \ne j$ consider the following integral $n \times n$-matrix

$$A_{ij}(\mathbf{v}) = \mathrm{diag}(v_1, ..., v_{j-1}, v_j - 1, v_{j+1}, ..., v_n) + E_{ij}.$$

Let $\tilde{M}^{ij} = \coprod_{\mathbf{v}\in\mathbf{V}} X^{(A_{ij}(\mathbf{v}))}$. The projections p_{ij} of the components give a map $\tilde{p}_{ij} : \tilde{M}^{ij} \to X$. For an open set $U \subset X$ and $f \in \mathcal{O}(U)$ we denote by $p_{ij}^* f$ the function $f \circ \tilde{p}_{ij}$ on $\tilde{p}_{ij}^{-1}(U)$ regarded as a function on the whole $\coprod_{A\in\mathbf{M}} U^{(A)} \subset \tilde{M}$ (so this function is zero on components not in $\tilde{p}_{ij}^{-1}(U)$). Similarly we denote by $p_{ii}^*(f)$ the function on $\coprod_{A\in\mathbf{M}} U^{(A)} \subset \tilde{M}$ which on components not of the form $U^{(\mathrm{diag}(\mathbf{v}))}$ is equal to 0 and on the component $U^{(\mathrm{diag}(\mathbf{v}))} = U^{(\mathbf{v})}$ is

$$p_{ii}^*(f)(I_1, ..., I_n) = \sum_{x\in I_i} f(x).$$

We denote also by $E_{ij}(f)$ the section $E_{ij} \otimes f$ of $\mathcal{G} = gl_n(\mathcal{O}_X)$.

(3.9) Theorem. *The assignment $E_{ij}(f) \mapsto p_{ij}^* f$ yields a Lie algebra homomorphism $\tau_U : \Gamma(U, \mathcal{G}) \to \Gamma(U^{(d)}, \tilde{\mathcal{G}})$ making the following diagram commute :*

$$\begin{array}{ccc} & \Gamma(U,\mathcal{G}) \stackrel{(3.3.1)}{\longrightarrow} & \Gamma(U^{(d)}, \mathcal{E}nd\, \mathcal{M}^{(\otimes d)}) \\ \tau_U & \downarrow & \nearrow (3.3.2) \\ & \Gamma(U^{(d)}, \tilde{\mathcal{G}}) & \end{array} .$$

Proof. Since the map $\Gamma(U^{(d)}, \tilde{\mathcal{G}}) \to \Gamma(U^{(d)}, \mathcal{E}nd\mathcal{M}^{(\otimes d)})$ is injective by (3.3), it is enough to show that the image of each $E_{ij}(f)$ under τ_U comes from an element of $\Gamma(U^{(d)}, \tilde{\mathcal{G}})$.

Consider the standard $gl_n(\mathbf{C})$-action on the space $\mathbf{C}^n$ with standard basis $e_1, ..., e_n$. The element E_{ij} takes $e_i \mapsto e_j$ and other base vectors to 0. Let I be any finite set, $|I| = d$, and let $\mathcal{O}(I)$ be the ring of functions $I \to \mathbf{C}$. The Lie algebra $gl_n(I) = gl_n(\mathbf{C}) \otimes \mathcal{O}(I)$ acts on the tensor product of I copies of $\mathbf{C}^n$, which we denote by $(\mathbf{C}^n)^{\otimes I}$. The natural basis in $(\mathbf{C}^n)^{\otimes I}$ is labelled by partitions $\mathbf{I} = (I_1, ..., I_n)$, $I_\nu \subset I$, $I = \coprod I_\nu$. More precisely, the basis vector corresponding to I is $e_{\mathbf{I}} = \bigotimes_{\nu \in I} e_{i(\nu)}$ where $i(\nu)$ is such that $\nu \in I_{i(\nu)}$. For $f \in \mathcal{O}(I)$ and distinct $i, j \in [n]$ the standard action of $E_{ij}(f) = E_{ij} \otimes f$ on $(\mathbf{C}^n)^{\otimes I}$ is immediately seen to be given by the formula

$$(3.10) \qquad e_{\mathbf{I}} \mapsto \sum_{\nu \in I_j} f(\nu) e_{(I_1, ..., I_i + \{\nu\}, ..., I_j - \{\nu\}, ..., I_n)}.$$

The assignment $E_{ij}(f) \mapsto p_{ij}^* f$ in Theorem 3.8 is nothing but the result of the simultaneous application of (3.10) to all subsets $I \subset U$, $|I| = d$. This completes the proof.

(3.11) Proof of Theorem 3.5. We first consider the case when the covering $Y \to X$ and the bundle $\mathcal{L}$ on Y are trivial, i.e., $Y = X \times [n]$, $\mathcal{L} = \mathcal{O}_Y$, so that we are in the situation of (3.7). Let $U \subset E$ be an affine open set. Then the space

$$\Gamma(U^{(d)}, \mathcal{G}^{(\oplus d)}) = \left(\bigoplus_{i=1}^{d} \Gamma(U^d, \mathrm{pr}_i^* \mathcal{G}) \right)^{S_d}$$

is generated, as a module over $\Gamma(U^{(d)}, \mathcal{O})$, by the image of $\Gamma(U, \mathcal{G})$ under the diagonal map, i.e., by elements of the form $(g(x_1), ..., g(x_n))$, $g \in \Gamma(U, \mathcal{G})$. We claim that the assignment $\tau_U : E_{ij}(f) \mapsto p_{ij}^* f$ of Theorem 3.8 extends, by $\mathcal{O}_{U^{(d)}}$-linearity, to a morphism $\Gamma(U^{(d)}, \mathcal{G}^{(\oplus d)}) \to \Gamma(U^{(d)}, \tilde{\mathcal{G}})$. Indeed, since $\mathcal{G} = \bigoplus_{i,j} \mathcal{O}_X \cdot E_{ij}$ as a coherent sheaf, we find that $\mathcal{G}^{(\oplus d)} = \bigoplus_{i,j} \mathcal{O}_X^{\oplus d} \cdot E_{ij}$. Thus it is enough to consider every $\mathcal{O}_X \cdot E_{ij}$ separately and to show that the following statement holds :

(3.12) Lemma. *Let $\phi_\alpha(x_1, ..., x_d) \in \mathcal{O}(U^d)$ be symmetric functions and $f_\alpha \in \Gamma(U, \mathcal{O})$ be such that*

$$\sum_\alpha \phi_\alpha(x_1, ..., x_d)(f_\alpha(x_1), ..., f_\alpha(x_d)) = 0 \quad \text{in} \quad \mathcal{O}(U^d) \oplus ... \oplus \mathcal{O}(U^d).$$

Then the section $\sum \phi_\alpha(x_1, ..., x_d) p_{ij}^* f_\alpha \in \Gamma(U^{(d)}, \tilde{\mathcal{G}})$ is equal to 0 for any i, j.

We leave the verification of this lemma to the reader. Theorem 3.5 follows in the case $Y = X \times [n]$, $\mathcal{L} = \mathcal{O}_Y$.

To treat the general case, let n be the degree (number of sheets) of the map $Y \to X$. Choose an etale covering $\{\psi_\alpha : U_\alpha \to X\}$ (so we have the maps $p_\alpha : \psi_\alpha^* Y = Y \times_X U_\alpha \to Y$) and choose trivializations $k_\alpha : u_\alpha \times [n] \to \psi_\alpha^* Y$ and $k_\alpha^* p_\alpha^* \mathcal{L} \to \mathcal{O}_{U_\alpha \times [n]}$. Then apply the assignments τ_{U_α} of Theorem 3.8 and check that they satisfy the descent condition to get a map defined over all the $X^{(d)}$. We leave the easy details to the reader.

(3.13) Algebra of convolution operators. We consider the situation of (3.8). Fix an open set $U \subset X$. Geiven a matrix $A \in \mathbf{M}$ we have the following diagdam, see (3.8.0):

$$X^{(\mathbf{v}^1)} \stackrel{pr_1}{\leftarrow} X^{(A)} \stackrel{pr_2}{\rightarrow} X^{(\mathbf{v}^2)}$$

To any function $f \in \mathcal{O}(U^{(A)})$ we associate the convolution operator

$$\Delta(A,f) : \mathcal{O}(U^{(\mathbf{v^1})}) \to \mathcal{O}(U^{(\mathbf{v^2})}) \quad , \quad \Delta(A,f) : \psi \mapsto (pr_2)_*((pr_1^*\psi)\cdot f)$$

Given a linear operator F of the form $F = \Delta(A,f) + \sum_i \Delta(B^i, g^i)$, we write $F = \Delta(A,f) + \cdots$, if all $B^i < A$ with respect to the Bruhat order (3.7).

We start by describing, in abstract terms, the composition of operations of type $\Delta(A,f)$. Let $\mathbf{T}$ be the set of 3-dimensional arrays $(t_{ijk})_{1\le i,j,k\le n}$ of non-negative integers, such that $\sum_{i,j,k} t_{ijk} = d$. For any $\mathbf{v}^1, \mathbf{v}^2, \mathbf{v}^3 \in \mathbf{V}$ put

$$\mathbf{T}(\mathbf{v}^1,\mathbf{v}^2,\mathbf{v}^3) = \Big\{ T \in \mathbf{T} : \sum_{j,k} t_{ijk} = v_i^1,\ \sum_{i,k} t_{ijk} = v_j^2,\ \sum_{i,j} t_{ijk} = v_k^3 \Big\}.$$

If $1 \le l \le m \le 3$ and $T \in \mathbf{T}(\mathbf{v}^1,\mathbf{v}^2,\mathbf{v}^3)$, let T^{lm} be the matrix in $\mathbf{M}(\mathbf{v}^l,\mathbf{v}^m)$ obtained by summing the entries of T along the indexed not named. For $T \in \mathbf{T}$ we set $X^{(T)} = \prod_{i,j,k} X^{(t_{ijk})}$ and write $pr_{lm} : X^{(T)} \to X^{(T^{lm})}$ for the natural projection. Given two matrices $A \in \mathbf{M}(\mathbf{v}^1,\mathbf{v}^2)$, and $B \in \mathbf{M}(\mathbf{v}^2,\mathbf{v}^3)$, put

$$\mathbf{T}(A,B) := \{T \in \mathbf{T} : T^{12} = A, T^{23} = B\} \qquad \mathbf{(3.13.0)}$$

The definition of the convolution gives (3.3):

(3.13.1) Lemma. *Let $A \in \mathbf{M}(\mathbf{v}^1,\mathbf{v}^2)$, and $B \in \mathbf{M}(\mathbf{v}^2,\mathbf{v}^3)$, $f \in \mathcal{O}(U^{(A)})$, $g \in \mathcal{O}(U^{(B)})$. Then, the composition of convolution operators on the LHS below is given by the following sum on the RHS:*

$$\Delta(A,f)\cdot\Delta(B,g) = \sum_{\{T\in\mathbf{T}(A,B)\}} \Delta\big(T^{13},\, pr_{13*}(pr_{12}^*(f)\cdot pr_{23}^*(g))\big).$$

(3.13.2) Corollary. *Suppose that, in the situation of Proposition 3.13.1, $A = \mathrm{diag}(d_1^1,...,d_n^1)$ is the diagonal matrix so that $\mathbf{v}^1 = \mathbf{v}^2$. Then we have*

$$\Delta(A,f)\,\Delta(B,g) = \Delta(B,h),$$

where $h = g\cdot p_2^* f$ and $p_2 : U^{(B)} \to U^{(\mathbf{v}^2)} = U^{(A)}$ is the natural projection.

Proof. The only array T which will contribute to the sum in (3.13.1), is given by

$$t_{ijk} = \begin{cases} 0, & i \ne j \\ b_{jk}, & i = j \end{cases}$$

The following result gives way to proving results by induction with respect to the Bruhat order.

(3.13.3) Proposition. *In the situation of Proposition 3.13.1, assume that $A = \mathrm{diag} + a_{p,p+1}E_{p,p+1}$ where 'diag' stands for diagonal matrix. Put $m = \max\{i : b_{p+1,i} \ne 0\}$, and suppose, in addition, that $b_{p+1,m} \ge a_{p,p+1}$. Then*

$$\Delta(A,f)\cdot\Delta(B,g) = \Delta(C,h) + ...,$$

where $C = B + a_{p,p+1}(E_{pm} - E_{p+1,m})$ and $h = pr_{13*}(pr_{12}^*(f)\cdot pr_{23}^*(g))$, the projections pr_{ij} corresponding to the array in (3.13.4).

Similarly, if $A - a_{p,p-1}E_{p,p-1}$ is diagonal, $m = \min\{i : b_{p-1,i} \ne 0\}$ and $b_{p-1,m} \ge a_{p,p-1}$, then we have a similar relation with $C = B + a_{p,p-1}(E_{pm} - E_{p-1,m})$ and the array T defined in an analogous way.

To prove the proposition, we need the following result.

(3.13.4) Lemma. *Let $A = \mathrm{diag} + a_{p,p+1}E_{p,p+1}$. For an arbitrary matrix B as above, the set $\mathbf{T}(A,B)$, see* (3.13.0)*, is in bijective correspondence with the following set of n-tuples*

$$\mathbf{S}(A,B) = \{s = (s_1, s_2, ..., s_n) \,|\, 0 \leq s_k \leq b_{p+1,k} \quad , \quad \sum_k s_k = a_{p,p+1}\}$$

To an n-tuple $s \subset \mathbf{S}(A,B)$ on assigns the following array $T(s) = (t_{ijk})$

$$\begin{cases} t_{iik} = b_{ik} \quad \text{for} \quad i \neq p+1 \\ t_{p+1,p+1,k} = b_{p+1,k} - s_k \\ t_{p,p+1,k} = s_k \\ t_{ijk} = 0 \quad \textit{otherwise}, \end{cases}$$

Moreover, we have $T(s)^{^{13}} = B + \sum_j s_j \cdot (E_{pj} - E_{p+1,j})$.

Proof of (3.13.4). The arrays $T = (t_{ijk})$ which will contribute to the sum in (3.13.1) are given by :

$$\begin{cases} \sum_k t_{ijk} = a_{ij} \\ \sum_i t_{ijk} = b_{jk}. \end{cases}$$

In particular,

$$\begin{cases} t_{ijk} = 0 \quad \text{if} \quad (i,j) \neq (i,i), (p,p+1) \\ t_{jjk} = b_{jk} \quad \text{if} \quad j \neq p+1 \\ t_{p,p+1,k} + t_{p+1,p+1,k} = b_{p+1,k}. \end{cases}$$

Thus T will have a non-zero contribution if and only if :

$$\begin{cases} t_{ijk} = 0 \quad \text{if} \quad (i,j) \neq (i,i), (p,p+1) \\ t_{jjk} = b_{jk} \quad \text{if} \quad j \neq p+1 \\ t_{p+1,p+1,k} = b_{p+1,k} - t_{p,p+1,k} \\ t_{p,p+1,k} \leq b_{p+1,k} \\ \sum_k t_{p,p+1,k} = a_{p,p+1} \\ \sum_k t_{p+1,p+1,k} = a_{p+1,p+1} \\ \sum_k b_{jk} = a_{jj} \quad \text{if} \quad j \neq p+1. \end{cases}$$

The last two equations follow directly from $\sum_j b_{ij} = d_i^2 = \sum_j a_{ji}$. The conditions above mean that there exists an n-tuple $s \subset \mathbf{S}(A,B)$ such that

$$\begin{cases} t_{iik} = b_{ik} \quad \text{for} \quad i \neq p+1 \\ t_{p+1,p+1,k} = b_{p+1,k} - s_k \\ t_{p,p+1,k} = t_k \\ t_{ijk} = 0 \quad \text{otherwise}. \end{cases}$$

The lemma follows.

Proof of (3.13.3). For $s \in \mathbf{S}(A,B)$ we put $C(s) := T(s)^{^{13}}$. Observe that, if $s, s' \in \mathbf{S}(A,B)$ then we have

$$C(s') \preceq C(s) \Leftrightarrow \forall j \quad , \quad \begin{cases} \sum_{r \geq j} s'_r \leq \sum_{r \geq j} s_r, \\ \sum_{r \leq j} s'_r \geq \sum_{r \leq j} s_r. \end{cases}$$

Thus, since $\sum_k s'_k = \sum_k s_k (= a_{p,p+1})$,

$$C(s') \preceq C(s) \Leftrightarrow \sum_{r \geq j} s'_r \leq \sum_{r \geq j} s_r \quad , \quad \forall j.$$

Observe, on the other hand that $b_{p+1,k} = 0$ if $k > m$. Hence $s_k = 0$ if $k > m$, for any $s \in \mathbf{S}(A,B)$. Since $b_{p+1,m} \geq a_{p,p+1}$, the set $\{T(s) \,|\, s \in \mathbf{S}(A,B)\}$ has a greatest element with respect to $\preceq$; it corresponds to the n-tuple $s = a_{p,p+1} \cdot (\delta_{1m}, \delta_{2m}, ..., \delta_{nm}) \in \mathbf{T}(A,B)$. The corresponding array looks like

$$T_{max} = \begin{cases} t_{ijk} = 0 \quad \text{if} \quad (i,j) \neq (i,i), (p,p+1) \\ t_{iik} = b_{ik} - \delta_{(i,k)\,(p+1,m)} a_{p,p+1} \\ t_{p,p+1,k} = \delta_{km} a_{p,p+1}. \end{cases}$$

Observe that the matix $C = \overset{13}{T}_{max}$ arising from this array is precisely the matrix $C = B + a_{p,p+1}(E_{pm} - E_{p+1,m})$. In particular, for any other matrix $C' \neq C$ arising on the RHS of the formula

$$\Delta(A,f) \cdot \Delta(B,g) = \sum_{C',h} \Delta(C',h) + ...,$$

we have $C' \preceq C$. Thus, proposition (3.13.3) is proved.

(3.14) Proof of Proposition 3.6 (b). We will prove by induction on $l(C) = \sum_{i \neq j} \binom{|i-j|+1}{2} c_{ij}$ that, for any $C \in \mathbf{M}(\mathbf{v}^1, \mathbf{v}^2)$ and $h \in \mathcal{O}(U^{(C)})$, the convolution operator $\Delta(C,h)$ belongs to $\mathrm{Im}\,\mathcal{U}(\tau_U)$. If $l(C) = 0, 1$ it is evident. Suppose $l(C) > 1$. Then C is not diagonal. Suppose for instance that C is not a lower-triangular matrix. Let (p,q) be the greatest element in $\{(i,j) \,|\, 1 \leq i < j \leq n \quad , \quad c_{ij} \neq 0\}$ with respect to the right-lexicographic order (by right-lexicographic order we mean $(i,j) > (p,q) \Leftrightarrow j > q \quad \text{or} \quad (j = q \quad \text{and} \quad i > p)$). Put

$$B = C + c_{pq}(E_{p+1,q} - E_{pq}) \quad \text{and} \quad A = \mathrm{diag}(v_1^1, v_2^1, ..., v_n^1) + c_{pq}(E_{p,p+1} - E_{p,p}).$$

Then $l(B) < l(C)$, $q = \max\{i \,|\, b_{p+1,i} \neq 0\}$,

$$b_{p+1,q} = c_{pq} + c_{p+1,q} \geq c_{pq} = a_{p,p+1},$$

and, for any $f \in \mathcal{O}(U^{(A)})$, $g \in \mathcal{O}(U^{(B)})$, (3.13.3) gives :

$$\Delta(A,f) \cdot \Delta(B,g) = \Delta(C,h) + ...$$

where $h = pr_{13*}(pr_{12}^*(f) \cdot pr_{23}^*(g))$, the projections pr_{ij} beeing computed with respect to the array T given in (3.13.6). In particular if $f = p_{p,p+1}^*(f_{p,p+1})$, $f_{p,p+1} \in \mathcal{O}(U^{(a_{p,p+1})})$, one gets :

$$h = p_{pq}^*(f_{p,p+1}) \cdot r^*(g) \in \mathcal{O}(U^{(C)}),$$

where $r \,:\, U^{(C)} \to U^{(B)}$ is given by :

$$\begin{cases} r_{|U^{(c_{ij})}} = id \,:\, U^{(c_{ij})} \to U^{(b_{ij})} & \text{if } (i,j) \neq (p,q), (p+1,q), \\ r_{|U^{(c_{pq})} \times U^{(c_{p+1,q})}} = \oplus \,:\, U^{(c_{pq})} \times U^{(c_{p+1,q})} \to U^{(b_{p+1,q})}. \end{cases}$$

It's then easy to conclude from the induction hypothesis since that $\Delta(A,f) \in \mathrm{Im}\,\mathcal{U}(\tau_U)$.

4. Elliptic cohomology construction of current algebras.

(4.1) The Steinberg variety. Fix $d, n \geq 0$. Let F be the variety of n-step flags in $\mathbf{C}^d$, i.e., filtrations $D = \{0 = D_0 \subseteq D_1 \subseteq D_2 \subseteq ... \subseteq D_n = \mathbf{C}^d\}$. The connected components of F are parametrized by partitions $\mathbf{v} = (v_1, ..., v_n)$ of d; the component $F_{\mathbf{v}}$ corresponding to $\mathbf{v}$ consisting of flags D such that $\dim D_i/D_{i-1} = v_i$. As in section (3.7), we denote the set of such partitions $\mathbf{V}$; we also conserve the other notations of that section.

Consider the group $GL_d(\mathbf{C})$ acting diagonally on $F \times F$.

(4.1.1) Proposition. *(a) The orbits of $GL_d(\mathbf{C})$ are parametrized by matrices $A \in \mathbf{M}$; the orbit O_A corresponding to A consists of pairs of flags (D, D') such that $\dim(D_i \cap D'_j) = a_{ij}$.*
(b) The closure of O_A contains O_B if and only if $B \leq A$ with respect to the Bruhat order on $\mathbf{M}$.

Denote by

$$Z \quad = \quad \bigcup_{A \in M} \overline{T^*_{O_A}(F \times F)} \quad \subset \quad T^*(F \times F) = T^*F \times T^*F$$

the union of the closures of the conormal bundles to O_A (the Steinberg- type variety). We regard Z as a correspondence from T^*F to T^*F. It is known that Z satisfies the following two properties :

(4.1.2) Both projections $Z \to T^*F$ are proper.

(4.1.3) The composition $Z \circ Z$ is equal to Z.

(4.2) Elliptic cohomology of the Steinberg variety. Fix an elliptic curve $E \to S$ over a base scheme S.

Put $G = U(d)$ and let $T \subset G$ be the subgroup of diagonal matrices. Denote by Ell_G, Ell_T the G-equivariant and T-equivariant elliptic cohomology theories with values in coherent sheaves on $\mathcal{X}_G = E^{(d)}$ and $\mathcal{X}_T = E^d$. We conserve all the other notations of Section 1.

We have an obvious G-action on Z and T^*F and the second projection $Z \to T^*F$ is G-equivariant (and proper). Thus Z is a Lagrangian correspondence between F and F, see (...) and so we have the microlocal Thom sheaf Ξ_Z on $Z_{\mathcal{X}_G}$. Since $Z \circ Z = Z$, it follows from Proposition (2.8.6) that the direct image ${\pi_Z}_*\Xi_Z$ is a sheaf of associative algebras on $\mathcal{X}_G$, and ${\pi_F}_*\Theta(T^*F)^{-1}$ is a sheaf of its modules. On the other hand, $\mathcal{X}_G = E^{(d)}$, and in (3.3) we constructed a sheaf of algebras $\tilde{\mathcal{G}} = \bigoplus_{A \in \mathbf{M}} {\pi_A}_* \mathcal{O}_{E^{(A)}}$ on $E^{(d)}$.

(4.3) Theorem. *There is an isomorphism of sheaves of algebras $\tilde{\mathcal{G}} \to {\pi_Z}_*\Xi_Z$.*

The rest of this section is devoted to the proof of Theorem 4.3.

(4.4) We start by considering closed orbits in $F \times F$. Fix $\mathbf{v} \in \mathbf{V}$ and $i, j \in \{1, ..., n\}$ such that $|i - j| \leq 1$. For such i, j let $A_{ij}(\mathbf{v})$ be the matrix introduced in (3.8). The orbit $O_{A_{ij}(\mathbf{v})}$ is the incidence correspondence in $F_\mathbf{v} \times F_{\mathbf{v}'}$, where $\mathbf{v}' = (v_1, ..., v_i + j - i, v_{i+1} - j + i, ..., v_n)$. We have thus the diagram of flag varieties

$$F_\mathbf{v} \longleftarrow O_{A_{ij}(\mathbf{v})} \longrightarrow F_{\mathbf{v}'}$$

and the result of application of the functor $\mathcal{X}_G$ to this diagram is just

$$E^{(\mathbf{v})} \leftarrow E^{(A_{ij}(\mathbf{v}))} \to E^{(\mathbf{v}')},$$

see (1.7.5). Denote for short

$$Z_{ij}(\mathbf{v}) = T^*_{O_{A_{ij}(\mathbf{v})}}(F_\mathbf{v} \times F_{\mathbf{v}'}).$$

This is an irreducible component of the "Steinberg-type" variety Z. Note that $T^*F_\mathbf{v}$, $T^*F_{\mathbf{v}'}$ and $Z_{ij}(\mathbf{v})$ are G-equivariantly homotopy equivalent to $F_\mathbf{v}$, $F_{\mathbf{v}'}$ and $O_{A_{ij}(\mathbf{v})}$ respectively; thus we can identify, in particular, $Z_{ij}(\mathbf{v})_{\mathcal{X}_G}$ with $E^{(A_{ij}(\mathbf{v}))}$. The right map in the diagram

$$T^*F_\mathbf{v} \longleftarrow Z_{ij}(\mathbf{v}) \longrightarrow T^*F_{\mathbf{v}'}$$

is proper and induces, therefore, the microlocal Thom sheaf $\Xi_{Z_{ij}(\mathbf{v})}$ on $E^{(A_{ij}(\mathbf{v}))}$. Proposition 2.8.6 implies the first part of the following assertion :

(4.5) Proposition. *(a) If $|i - j| \leq 1$, the sheaf $\Xi_{Z_{ij}(\mathbf{v})}$ is naturally isomorphic to $\mathcal{O}_{E^{(A_{ij}(\mathbf{v}))}}$.*
(b) The sheaf $\Theta(T^*F)^{-1}$ on $F_{\mathcal{X}_G} = \coprod_{\mathbf{v} \in \mathbf{V}} E^{(\mathbf{v})}$ consists of functions $g(I_1, ..., I_n)$, $I_\nu \in E^{(v_\nu)}$ which are allowed poles of at most 1-st order along the locus of $(I_1, ..., I_n)$ such that $I_\mu \cap I_\nu \neq \emptyset$ for some $\mu \neq \nu$.
(c) If $|i - j| \leq 1$, the map

$${\pi_{Z_{ij}(\mathbf{v})}}_* \Xi_{Z_{ij}(\mathbf{v})} \to \mathcal{H}om({\pi_{F_\mathbf{v}}}_* \Theta(T^*F_\mathbf{v})^{-1}, {\pi_{F_{\mathbf{v}'}}}_* \Theta(T^*F_{\mathbf{v}'})^{-1})$$

from (2.7.1) associates to $f \in \mathcal{O}(U^{(A_{ij}(\mathbf{v}))})$, $U \subset E$, the convolution operator $\Delta(A_{ij}(\mathbf{v}), f)$ from (3.13).

Proof. (b) Denote by $\mathcal{D}_1 \subset ... \subset \mathcal{D}_n$ the tautological flag of bundles on F. It is well known that TF has a G-equivariant filtration whose quotients are $\mathcal{H}om(\mathcal{D}_i/\mathcal{D}_{i-1}, \mathcal{D}_j/\mathcal{D}_{j-1})$, $i < j$. Our statement follows from this and the definition of Θ. Part (c) follows from the localisation formula given in (2.9) (see [V]).

(4.6) The rest of the proof of (4.3) is done by induction, similarly to the reasoning of (3.13), from which we freely borrow the notation. We use, in particular, the Bruhat order $\leq$ and the binary operation $*$ on the set $\mathbf{M}$. For $A \in \mathbf{M}$, an open $U \subset E$ and $f \in \mathcal{O}(U^{(A)})$ we denote $\Delta(A, f)$ the section of $\tilde{\mathcal{G}}$ over $U^{(d)}$ corresponding to f. We denote by $\tilde{\mathcal{G}}_{\leq A} = \bigoplus_{B \leq A} \pi_{B*}\mathcal{O}_{E^{(B)}}$ the subsheaf in $\tilde{\mathcal{G}}$ spanned (over $\mathcal{O}_{E^{(d)}}$) by sections of the form $\Delta(B, f)$ where $B \leq A$. Then $\tilde{\mathcal{G}}_{\leq A} \cdot \tilde{\mathcal{G}}_{\leq B} \subset \tilde{\mathcal{G}}_{\leq A*B}$. So we have a filtration of the sheaf of algebras $\tilde{\mathcal{G}}$ labelled by $\mathbf{M}$.

On the other hand, let us denote the sheaf $\pi_{Z*}\Xi_Z$ on $E^{(d)}$ by $\widehat{\mathcal{G}}$. The Bruhat order, $\leq$, on $\mathbf{M}$ is, in geometric terms, just the inclusion of orbit closures in $F \times F$ (i.e. $\overline{O_A} = \bigcup_{B \leq A} O_B$). Let

$$Z_{\leq A} = \bigcup_{B \leq A} T^*_{O_B}(F \times F),$$

and $Z_{<A}$ be the similar union over $B < A$. The complement $Z_A = Z_{\leq A} \setminus Z_{<A} = T^*_{O_A}(F \times F)$ is a smooth locally-closed G-sub-variety of $T^*(F \times F)$ paved by complex affine spaces. Let $i_{\leq A}$ be the inclusion maps of $Z_{\leq A}$ in Z. By induction, using the long exact sequence in "homology" (2.2.1) and the vanishing of $\mathrm{Ell}^{2k+1}_G(\mathbf{C}^n)$, we get that the Gysin map (see (2.8.8.2))

$$i_{\leq A*} : \pi_{Z_{\leq A}*}\Xi_{Z_{\leq A}} \to \pi_{Z*}\Xi_Z = \widehat{\mathcal{G}}$$

is an injective homomorphism of coherent sheaves on $E^{(d)}$. Let $\widehat{\mathcal{G}}_{\leq A}$ be its image. Then $\widehat{\mathcal{G}}_{\leq A} \cdot \widehat{\mathcal{G}}_{\leq B} \subset \widehat{\mathcal{G}}_{\leq A*B}$. We have thus constructed a filtration of the sheaf of algebras $\widehat{\mathcal{G}}$ labelled by $\mathbf{M}$.

(4.6.1) Lemma. *The sheaf $\widehat{\mathcal{G}}$ is locally free. Moreover, the morphism $\Psi : \widehat{\mathcal{G}} \to \mathcal{E}nd\big(\pi_{F*}\Theta(T^*F)^{-1}\big)$ induced by the action of $\widehat{\mathcal{G}}$ on $\pi_{F*}\Theta(T^*F)^{-1}$ is injective.*

Proof. The long exact sequence (2.2.1) gives short exact sequences

$$0 \to \widehat{\mathcal{G}}_{<A} \to \widehat{\mathcal{G}}_{\leq A} \to \pi_{Z_A*}\Xi_{Z_A} \to 0,$$

where Ξ_{Z_A} is defined as in (2.8.8.1). Since $\Xi_{Z_A} \simeq \mathcal{O}_{Z_{A\mathcal{X}_G}} \simeq \mathcal{O}_{E^{(A)}}$, an induction on $A \in \mathbf{M}$ proves that $\widehat{\mathcal{G}}$ is locally free. Set $E^{(d)}_{reg} = E^{(d)} \setminus \bigcup_{H \subset U(d)} \mathcal{X}_H$, the union beeing taken over all the closed proper subgroups of $U(d)$. For any regular element $g \in U(d)$, the morphism of localisation of correspondences (see (2.10.3))

$$\rho_g : \pi_{Z*}\Xi_Z \to \pi_{Z^g*}\Xi_{Z^g}$$

is an isomorphism of sheaves of algebras over $E^{(d)}_{reg}$. But Z^g is formed of a finite number of points of Z labelled by $\mathbf{M}$, and one can thus easily see that the map Ψ is an isomorphism over $E^{(d)}_{reg} \subset E^{(d)}$. Since $\widehat{\mathcal{G}}$ is locally free, Ψ is injective.

We now proceed to construct a homomorphism of sheaves of algebras $\Phi : \tilde{\mathcal{G}} \to \widehat{\mathcal{G}}$. Namely, from (3.14) the sheaf of algebras $\tilde{\mathcal{G}}$ is generated by its subsheaves $\tilde{\mathcal{G}}_{\leq A_{ij}(\mathbf{v})}$ for $\mathbf{v} \in \mathbf{V}$ and $|i - j| \leq 1$. Such matrices $A_{ij}(\mathbf{v})$ are minimal with respect to $\leq$, since the corresponding orbits are closed. Thus the arguments of (4.4) give identifications $\Phi_{ij\mathbf{v}} : \tilde{\mathcal{G}}_{\leq A_{ij}(\mathbf{v})} \to \widehat{\mathcal{G}}_{\leq A_{ij}(\mathbf{v})}$.

(4.6.2) Lemma. *The isomorphisms $\Phi_{ij\mathbf{v}}$ extend to a unique homomorphism $\Phi : \tilde{\mathcal{G}} \to \widehat{\mathcal{G}}$ of sheaves of algebras such that $\Phi(\tilde{\mathcal{G}}_{\leq A}) \subset \widehat{\mathcal{G}}_{\leq A}$ for any A.*

Proof. The sheaf $\pi_{F*}\Theta(T^*F)^{-1}$ is locally free, since it is a pushdown of a line bundle with respect to the finite flat map $\pi_F : F_{\mathcal{X}_G} = \coprod_{\mathbf{v}} E^{(\mathbf{v})} \to E^{(d)}$. Therefore the sheaf of algebras $\mathcal{E}nd\big(\pi_{F*}\Theta(T^*F)^{-1}\big)$ on $E^{(d)}$ is locally free as well.

Let $\mathbf{R}$ be the algebra formed by operators $\Delta(A, f)$ where $A \in \mathbf{M}$, and f is a rational function on $E^{(A)}$. Denote $\mathcal{R}$ the constant sheaf on $E^{(d)}$ with stalk $\mathbf{R}$. We claim that $\mathcal{E}nd\big(\pi_{F*}\Theta(T^*F)^{-1}\big)$ is naturally a subsheaf in $\mathcal{R}$. Indeed, let $E^{(d)}_{gen} \subset E^{(d)}$ be the open set formed by unordered tuples of *distinct* points. Then the map π_F is unramified over $E^{(d)}_{gen}$. Moreover, the line bundle $\Theta(T^*F)^{-1}$ is trivialized over $\pi_F^{-1}(E^{(d)}_{gen})$ by Proposition 4.5(b). This gives an identification of $\tilde{\mathcal{G}}$ with $\mathcal{E}nd\big(\pi_{F*}\Theta(T^*F)^{-1}\big)$ over $E^{(d)}_{gen}$. This, in virtue of the local freeness of $\mathcal{E}nd\big(\pi_{F*}\Theta(T^*F)^{-1}\big)$, implies our claim.

By definition, $\tilde{\mathcal{G}}$ injects in $\mathcal{R}$. Consider the diagram

$$\textbf{(4.6.3)} \qquad \begin{array}{ccccc} & & \mathcal{R} & & \\ & \nearrow & & \nwarrow & \\ \widehat{\mathcal{G}} & & & & \tilde{\mathcal{G}} \\ \uparrow & & & & \uparrow \\ \widehat{\mathcal{G}}_{\leq A_{ij}(\mathbf{v})} & & \xleftarrow{\Phi_{ij\mathbf{v}}} & & \tilde{\mathcal{G}}_{\leq A_{ij}(\mathbf{v})} \end{array}$$

which is commutative by Proposition 4.5 (c). To show the existence of Φ, it is enough to verify that the $\Phi_{ij\mathbf{v}}$ preserve the relations between sections of $\tilde{\mathcal{G}}_{\leq A_{ij}(\mathbf{v})}$ which (see Proposition 3.6 (b)), generate $\tilde{\mathcal{G}}$ as an algebra. More precisely, we need to prove that for any sections s_k of $\tilde{\mathcal{G}}_{\leq A_{i_k,j_k}(\mathbf{v}_k)}$, $k = 1, ..., r$, and any relation $F(s_1, ..., s_r) = 0$ holding in $\tilde{\mathcal{G}}$, we have $F(\Phi(s_1), ..., \Phi(s_r)) = 0$ (we have omitted the subscripts in the Φ's). But this is a consequence of the commutativity of the diagrams (4.6.3) and of the injectivity of the map $\tilde{\mathcal{G}} \to \mathcal{R}$. The compatibility of Φ with the filtrations is immediate since the filtrations on $\tilde{\mathcal{G}}$ and $\widehat{\mathcal{G}}$ are compatible with the product. Lemma 4.6.2 is proved.

(4.6.4) Lemma. *For any $A \in \mathbf{M}$ the homomorphism Φ induces isomorphisms $\tilde{\mathcal{G}}_{\leq A}/\tilde{\mathcal{G}}_{<A} \xrightarrow{\sim} \widehat{\mathcal{G}}_{\leq A}/\widehat{\mathcal{G}}_{<A}$.*

Proof. As in the proof of Lemma 4.6.2, (2.2.1) and (2.8.8.1) imply that $\widehat{\mathcal{G}}_{\leq A}/\widehat{\mathcal{G}}_{<A} \simeq \mathcal{O}_{E^{(A)}}$. By construction, $\tilde{\mathcal{G}}_{\leq A}/\tilde{\mathcal{G}}_{<A} = \mathcal{O}_{E^{(A)}}$. Thus, the induced map $\Phi_A : \tilde{\mathcal{G}}_{\leq A}/\tilde{\mathcal{G}}_{<A} \xrightarrow{\sim} \widehat{\mathcal{G}}_{\leq A}/\widehat{\mathcal{G}}_{<A}$ is identified with an endomorphism of the simple sheaf $\mathcal{O}_{E^{(A)}}$. Moreover, Φ_A is non-zero since by localisation of correspondences it is an isomorphism over $E^{(d)}_{reg} \subset E^{(d)}$ (see the proof of (4.6.1)). Thus the map Φ_A is bijective.

Lemma 4.6.4 implies that Φ is an isomorphism.

5. Classical elliptic algebras and elliptic cohomology.

(5.1) Heisenberg groups. We start with recalling some well known facts [Mum]. Let $E \to S$ be an elliptic curve over a base scheme S. For any $n \geq 1$ we denote by $n_E : E \to E$ the homomorphism of multiplication by n. Let $E_n = \mathrm{Ker}\, n_E \subset E$ be the group of points of order n, and let $\mu_n \subset \mathbf{C}^*$ be the group of nth roots of 1. There is a canonical non-degenerate skew-symmetric pairing

$$(\alpha, \beta) \mapsto \langle \alpha, \beta \rangle, \quad E_n \otimes_{\mathbf{Z}} E_n \to \mu_n$$

called the Weil pairing. We will sometimes use the notation $\langle \alpha, \beta \rangle_n$ to indicate the dependence on n. For instance, if $m = nc$ is a multiple of n, then $E_n \subset E_m$, and for any $\alpha, \beta \in E_n$ we have $\langle \alpha, \beta \rangle_m = \langle \alpha, \beta \rangle_n^c$.

Fix $n \geq 1$. The pairing $\langle \alpha, \beta \rangle$ defines a central extension

$$1 \to \mu_n \to H_n \to E_n \to 1$$

known as the Heisenberg group. As a set, $H_n = E_n \times \mu_n$, with multiplication $(\alpha, \zeta) \cdot (\beta, \xi) = (\alpha + \beta,\ \langle \alpha, \beta \rangle \zeta \xi)$.

(5.2) The Heisenberg representation. A representation T of H in a vector space (or, more generally, an action on any variety acted upon by $\mathbf{C}^*$) is said to have central charge c, if for $\zeta \in \mu_n$ we have $T(\zeta) = \zeta^c \cdot \mathrm{Id}$. For any $c \in \mathbf{Z}/n$ relatively prime to n there is a unique, up to isomorphism, n-dimensional irreducible representation H_n with central charge c. We denote it $T_c : H_n \to \mathrm{Aut}(M_c)$. A particular model for M_c can be obtained as follows. Choose a subgroup $\Gamma \subset E_n$, $\Gamma \simeq \mathbf{Z}/n$ and define the space

$$M_c(\Gamma) = \big\{ f : E_n \to \mathbf{C} \,|\, f(\alpha+\gamma) = \langle \alpha, \gamma \rangle^c f(\alpha) \quad \forall \alpha \in E_n, \gamma \in \Gamma \big\}.$$

The action of $(\beta, \zeta) \in H_n$ on $M_c(\Gamma)$ is given by $T_c(\beta,\zeta)(f)(\alpha) = \zeta^c \langle \alpha, \beta \rangle f(\alpha+\beta)$.

If we choose a complex uniformization $E = \mathbf{C}/\mathbf{Z} \oplus \mathbf{Z}\tau$, $\mathrm{Im}(\tau) > 0$, then E_n is identified with $(\mathbf{Z}/n)^2$ and the Weil pairing has the form

$$\langle \alpha, \beta \rangle = \exp\bigg(\frac{2\pi i}{n}(\alpha_1\beta_2 - \alpha_2\beta_1) \bigg), \quad \alpha_i, \beta_i \in \mathbf{Z}/n. \tag{5.2.1}$$

The space $M_c(\Gamma)$ for $\Gamma = \mathbf{Z}/n \oplus 0$, is identified with $\mathbf{C}^n$ and the representation T_c sends $(\alpha, \zeta) \in H_n$ into $\zeta^c I_{10}^{c\alpha_1} I_{01}^{\alpha_2}$, where

$$I_{10} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \omega & 0 & \dots & 0 \\ 0 & 0 & \omega^2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & \omega^{n-1} \end{pmatrix}, \quad I_{01} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}, \quad \omega = e^{\frac{2\pi i}{n}}. \tag{5.2.2}$$

For $x \in E$ we denote $t_x : E \to E$ the translation by x. The main reason for using Heisenberg groups is the following :

(5.3) Proposition. *Let L be a line bundle of degree n on E. Then for any $\alpha \in E_n$ we have $t_\alpha^* L \simeq L$. Moreover, there is an action of H_n in (the total space of) L extending the action of E_n on E by translations. This action has central charge 1. The resulting representation of H_n on the space $\Gamma(E, L)$ is irreducible, i.e. $\Gamma(E,L) \simeq M_1$.*

By comparing Weil pairings on E_n and E_m, $n|m$, we deduce the next statement :

(5.4) Corollary. *If L is a line bundle on E of degree nc, then, for any E_n-invariant open set $U \subset E$, the space $\Gamma(U, L)$ is a representation (possibly reducible) of H_n with central charge c.*

(5.5) Vector bundles over an elliptic curve. Let us fix c such that $(c,n) = 1$. Let $\mathcal{M}_E(n,c)$ be the set of indecomposable rank n vector bundles on E with $c_1(V) = c$. It was proved by Atiyah [A1] that :

(5.5.1) Any two bundles $V, V' \in \mathcal{M}_E(n,c)$ are related by $V' = V \otimes L$ where L is a line bundle of degree 0.

(5.5.2) Any $V \in \mathcal{M}_E(n,c)$ is simple, i.e. $H^0(E, \mathcal{E}nd(V)) = \mathbf{C}$. Moreover, $H^1(E, \mathcal{E}nd(V)) = 0$.

Let us recall, for future use, three equivalent constructions of bundles from $\mathcal{M}_E(n,c)$. Each of them gives all the bundles.

(5.6) The Heisenberg construction. Choose a line bundle L on E of degree nc. By Corollary 5.4, the direct image $n_{E*}L$ has an action of H_n in each fiber, with central charge c. Put $V = \mathrm{Hom}_{H_n}(n_{E*}L, M_c)$ to be the vector bundle of spaces of multiplicities of M_c in these representations. Then $V \in \mathcal{M}_E(n,c)$.

(5.7) The direct image construction. Let $\Gamma \subset E_n$ be a subgroup isomorphic to $\mathbf{Z}/n$. We have a commutative diagram

$$\begin{tikzcd} E \arrow[r, "q"] \arrow[dr, "n_E"'] & E/\Gamma \arrow[d, "p"] \\ & E \end{tikzcd} \tag{5.7.1}$$

where p is an isogeny of degree n with $\mathrm{Ker}(p) = E_n/\Gamma$. Then any $V \in \mathcal{M}_E(n,c)$ can be obtained as p_*L where L is a line bundle on E/Γ of degree c.

Note that $\tilde{L} = q^*L$ is a line bundle on E of degree nc, and $q_*\tilde{L}$ is canonically identified with $L \otimes M_c(\Gamma)$. This shows that

$$n_{E*}\tilde{L} = p_*q_*\tilde{L} = p_*L \otimes M_c(\Gamma)$$

and thus the constructions (5.7) and (5.6) are equivalent.

(5.8) The Fourier transform construction. Let P be the Poincaré line bundle on $E \times E$ (it corresponds to the divisor $\Delta - (E \times \{0\}) - (\{0\} \times E)$ where Δ is the diagonal). Let $p_1, p_2 : E \times E \to E$ be the projections. Then any $V \in \mathcal{M}_E(n,c)$ can be obtained as

$$V = R^0 p_{2*}\big(p_1^* L \otimes P^{\otimes c}\big), \qquad (\mathbf{5.8.1})$$

for a line bundle L on E of degree n. For the case $c = 1$ this construction is precisely the Fourier-Mukai transform of L, see [Muk].

The construction (5.7) can be found in [Od]. The equivalence of (5.7) and (5.6) was already explained. To see the equivalence with (5.8), note that each fiber of the RHS of (5.8.1) is the space of sections of a line bundle of degree n on E so has a Heisenberg group action. Taking eigenspaces of a generator of $\Gamma \subset E_n$ we split the fiber into a direct sum of 1-dimensional vector subspaces. This means that V is the pushdown of a line bundle from an unramified covering. We leave the details to the reader.

By (5.5.1), the sheaf $\mathcal{E}nd(V)$ for $V \in \mathcal{M}_E(n,c)$ depends only on n, c. We denote this sheaf by $\mathcal{G}_{n,c}$ or simply $\mathcal{G}$.

(5.9) Proposition. *(a) As a vector bundle, $\mathcal{G}_{n,c} = \bigoplus_{\alpha \in E_n} L_\alpha$, where $L_\alpha = \mathcal{O}_E(0 - \alpha)$ is the line bundle of degree 0 corresponding to α. The multiplication $m : \mathcal{G}_{n,c} \otimes \mathcal{G}_{n,c} \to \mathcal{G}_{n,c}$ restricts to isomorphisms $m_{\alpha\beta} : L_\alpha \otimes L_\beta \to L_{\alpha+\beta}$ such that $m_{\beta\alpha} = \langle \alpha, \beta \rangle^c \cdot m_{\alpha\beta}$.*
(b) The pullback $n_E^* \mathcal{G}_{n,c}$ (where $n_E : E \to E$ is the multiplication by n) is identified, as a sheaf of algebras, with $\mathrm{Mat}_n(\mathcal{O}_E)$. More precisely, for an open $U \subset E$ the space $\Gamma(U, \mathcal{G}_{n,c})$ is identified with the space of matrix functions $A(x) \in \mathrm{Mat}_n(\mathcal{O}(n_E^{-1}(U)))$ satisfying the condition

$$A(x + \alpha) = T_c(\alpha) A(x) T_c(\alpha)^{-1}, \quad \forall \alpha \in E_n. \qquad (\mathbf{5.9.1})$$

Part (a) can be found in [A1]. Part (b) follows at once from the Heisenberg construction of V. We will call (5.9.1) the automorphy condition, see [RS].

(5.10) Classical elliptic algebras. Fix a finite subset $P \subset E$. The classical elliptic algebra $\mathbf{el}_{n,c}$ is, by definition, the Lie algebra of global sections $\Gamma(E \backslash P, \mathcal{G}_{n,c})$, where $\mathcal{G}_{n,c} = \mathcal{E}nd(E)$ for any bundle $V \in \mathcal{M}_E(n,c)$. Let also $\mathcal{SG}_{n,c} = \mathbf{sl}(V)$ be the subsheaf of traceless endomorphisms and $\mathbf{sel}_{n,c}$ be the Lie algebra of its global sections over $E \setminus P$. By (5.5.2) the sheaf $\mathcal{SG}_{n,c}$ has both H^0 and H^1 vanishing. This makes the subalgebra $\mathbf{sel}_{n,c}$ into a Lie bialgebra by the procedure described in [Dr1], [C]. The Lie bialgebra structure is the map

$$\delta : \mathbf{sel}_{n,c} \to \wedge^2 \mathbf{sel}_{n,c} \subset \mathbf{sel}_{n,c} \otimes \mathbf{sel}_{n,c}, \quad \delta(x) = [x \otimes 1 + 1 \otimes x, r_{n,c}],$$

where $r_{n,c} \in \mathbf{sel}_{n,c} \hat{\otimes} \mathbf{sel}_{n,c}$ is the classical elliptic r-matrix ($\hat{\otimes}$ means completed tensor product). Let us describe $r_{n,c}$ explicitly. In the language of Proposition 5.9 (b), this is the meromorphic function $r_{n,c} : E \times E \to gl_n \otimes gl_n$ given by

$$r_{n,c}(u,v) = \sum_{\alpha \in E_n - \{0\}} w_\alpha(u - v) T_c(\alpha) \otimes T_c(\alpha)^{-1},$$

where w_α, $\alpha \in E_n$ are meromorphic functions on E uniquely characterized by the following conditions (see [B] for an explicit formula) :
(1) $w_\alpha(u + \beta) = \langle \beta, \alpha \rangle w_\alpha(u), \quad \forall \alpha, \beta \in E_n$.
(2) $w_0 \cong 1$ and if $\alpha \neq 0$, then w_α is regular on $E \backslash E_n$ with simple poles at points of E_n and $\mathrm{res}_{u=0} w_\alpha(u) = 1$.

When $c = 1$, the matrix $r_{n,1}$ was found by Belavin [B]. The possibility of considering any c prime to n was pointed out by Cherednik.

(5.11) Automorphy conditions and correspondences. We now proceed to construct a realization of the sheaf of Lie algebras $\mathcal{G} = \mathcal{G}_{n,c}$ by correspondences, by using the description of $\mathcal{G}$ given by Proposition 5.9 (b). We suppose that there is a fixed level n structure on E, i.e. an identification $E_n \xrightarrow{\sim} (\mathbf{Z}/n)^2 \times S$ such that the Weil pairing has the form (5.2.1). Thus the representation T_c is given by the matrices (5.2.2). We denote $\Gamma = \mathbf{Z}/n \oplus 0$.

We assume the framework and notation of (2.7) (specialized to our case). Let us identify the set $[n]$ with $\mathbf{Z}/n$, so $\mathbf{Z}/n$ acts on $[n]$ by translation. Consider the action of E_n on $E \times [n]^2$ given by

$$(\alpha_1, \alpha_2) \cdot (x, b, c) = (x + \alpha_1, b + \alpha_2, c - \alpha_2).$$

We extend this action first to the variety

$$\tilde{M} = (E \times [n]^2)^{(d)} = \bigoplus_{A \in \mathbf{M}} E^{(A)}.$$

Second, we extend the action to an action on the sheaf $\mathcal{O}_{\tilde{M}}$ by setting

$$(\alpha \cdot f)(z) = \omega^{\sum_{i,j} c\alpha_1 a_{ij}(i-j)} f(\alpha \cdot z), \quad z \in \tilde{M}. \tag{5.12}$$

Let

$$\mathrm{nor} : \tilde{M} \to M = (E \times [n])^{(d)} \times_{E^{(d)}} (E \times [n])^{(d)}$$

be the natural projection (normalization morphism, see (2.7)).

Recall (Theorem 3.3) that the variety M is nothing but $Z_{\mathcal{X}_{U(d)}}$, the spectrum of the $U(d)$ -equivariant elliptic cohomology of the Steinberg variety Z, and the sheaf $\mathrm{nor}_* \mathcal{O}_{\tilde{M}}$ is the same as Ξ_Z, the microlocal Thom sheaf of the second projection $Z \to F$. Recall also that $\pi_Z : Z_{\mathcal{X}_{U(d)}} \to \mathcal{X}_{U(d)} = E^{(d)}$ is the natural projection.

Set $U = n_E^{-1}(E \backslash P)$. Formula (5.12) defines an action of E_n in the space $\Gamma(U^{(d)}, \pi_{Z*}\Xi_Z)$. In the previous section we have constructed a homomorphism of Lie algebras $\Psi : \Gamma(U, gl_n(\mathcal{O})) \to \Gamma(U^{(d)}, \pi_{Z*}\Xi_Z)$.

(5.13) Theorem. *The map Ψ restricts to a surjective algebra homomorphism*

$$\mathcal{U}(\mathbf{el}_{n,c}) \to \Gamma(U^{(d)}, \pi_{Z*}\Xi_Z)^{E_n}.$$

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